Differential eqⁿs

Ordinary Differential eqⁿ

$$y^{(n)} = f(t, y, y', y'', \dots y^{n-1})$$

$\uparrow$ independent variable

all other are dependent variable.

Order of an ODE is the highest derivative.

System of DEs:

ex: SIR model

$$S' = -aSI$$
$$I' = aSI - bI$$
$$R' = bI$$

ex: $y'' - 4y' + 3y = 0$

$y_1(t) = e^t$ solves this.

$e^t - 4e^t + 3e^t = 0$

$y_1(t) = c_1 e^t$ solves this

$y_2(t) = c_2 e^{3t}$ also solves this.

$c_2\, 9e^{3t} - 4c_2\, 3e^{3t} + 3c_2 e^{3t} = 0.$

$y(t) = c_1 e^t + c_2 e^{3t}$ also solves this.

$y(t)$ is the general solⁿ i.e. contains all possible Solutions.

An ODE $y^n = f(t, y, y', \dots, y^{n-1})$ together with initial conditions $y(0) = y_0$, $y'(0) = y_0'$, … $y^{n-1}(0) = y_0^{n-1}$ makes an IVP.

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Separation of variables

Exponential Growth:

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int K\,dt$$

$$\ln|y| = Kt + c$$

$$e^{\ln|y|} = e^{Kt+c}$$

$$|y| = e^{Kt} \cdot e^{c} = \tilde{c}\, e^{Kt}$$

$$y = ce^{Kt}$$

Generically

$$\cancel{\frac{dy}{dt}} = / \frac{dy}{dt} = f(t)g(y)$$

$$\frac{dy}{g(y)} = f(t)dt$$

$$\int \frac{dy}{g(y)} = \int f(t)\, dt$$

ex: $$\frac{dy}{dx} = \frac{xy}{y^2+1}$$

$$= \quad \frac{y^2+1}{y}\, dy = x\,dx$$

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + c$$

It is an implicit solution.

There also is a solution: Singular solution: $y=0$.

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Newton's Law of Cooling

Ambient Temp $A = 68°F$

$T(0) = 161°F$

$T(2) =$ ~~153°F~~ $153.7°F$

$$\frac{dT}{dt} = -K(T-A)$$

$$\int \frac{dT}{T-A} = \int -K\,dt$$

$$\frac{1\,dT}{T-A} = -Kt \Rightarrow \ln|T-A| = -Kt + C$$

$$T = A + De^{-Kt}$$

$A = 68°F$

$t = 0, \quad T = 161$

$161 - 68 = D \Rightarrow D = 93.$

We can get K using our second condition.

$t = 2, \quad T = 153$

$$153.7 = 68 + 93e^{-2K}$$

$$85.7/93 = e^{-2K}$$

$$\ln(85.7/93) = -2K$$

$$K = -\frac{1}{2}\ln\left(\frac{85.7}{93}\right)$$

$K = 0.0409.$

$$T = 68 + 93e^{-0.0409t}$$

Different scenarios have different values of K.

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Geometric meaning of DE

$\frac{dy}{dx} = f(x,y)$ ⟷ slope field

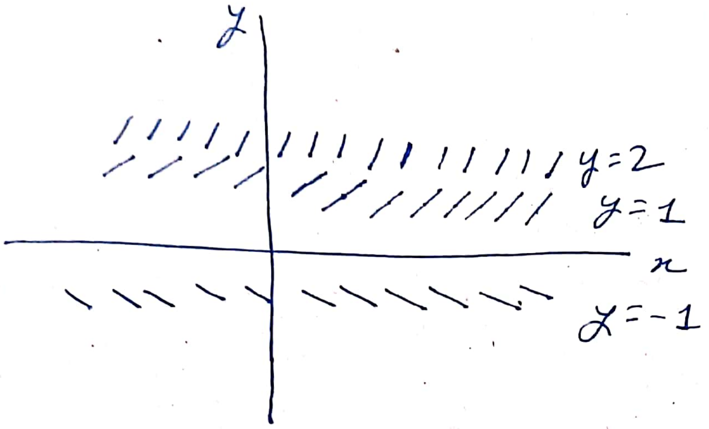
↓

slope

IC: $y(x_0) = y_0$ ⟷ Point (x_0, y_0)

Solution $y(x)$ ⟷ Integral curve.

$dy/dx = y$



Isocline : A curve $f(x,y) = c$.

'Solution solves the ODE';
'Integral curve tangent to the slope field'.

Existence & Uniqueness

Given an IVP

$y' = f(x,y)$, $y(x_0) = y_0$.

- Does a soln exist?
- Is that solution unique?

ex: $xy' = 1$, $y(0) = 0$.

Rewrite $y' = \frac{1}{x}$

It is not defined at $x = 0$.

No solution to this IVP.

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ex: $y' = y^{1/3}$, $y(0) = 0$.

$y = 0$ is a solution.

Separate variables

$$\int y^{-1/3} dy = \int dx$$

$$\frac{3}{2} y^{2/3} ~~+~~ = x + c$$

$\therefore$ IC: $0 = 0 + c$

$$\frac{3}{2} y^{2/3} = x$$

$$y = \pm \left(\frac{2}{3} x\right)^{3/2}.$$

we have 2 different solution. & also

$y = 0$ also works.

Note:

$$\frac{\partial}{\partial y} y^{1/3} = \frac{y^{-2/3}}{3}$$

This is undefined at $y(0) = 0$.

Theorem: If ~~y~~ f & $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0) then there is a unique solution on an interval $\alpha < x_0 < \beta$ to the IVP.

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

Remark: f continuous guarantees existence only.

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Linear DE & Integrating factors

$$x^2 y'' + \sin(x) y' + 3y = e^x$$

↓

Linear second order differential equation.

An ODE is linear if

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_0(x) y = b(x).$$

A Linear ODE is Homogeneous if

$$b(x) = 0.$$

* Standard form of 1st order Linea:

$$y' + p(x) y = f(x)$$

Integrating factor method

Multiply by $\gamma(x)$.

$$\gamma(x) y' + \gamma(x) p(x) y = \gamma(x) f(x)$$

$$\overset{?}{=}$$

$$\frac{d}{dx}(\gamma(x) y)$$

$$= \gamma(x) y' + \gamma'(x) y.$$

$$\Rightarrow \quad \gamma'(x) = \gamma(x) p(x).$$

$$\int \frac{\gamma'(x)}{\gamma(x)} dx = \int p(x) dx$$

$$\ln(\gamma(x)) = \int p(x) dx$$

$$\boxed{\gamma(x) = e^{\int p(x) dx}}$$

$$\frac{d}{dx}(\gamma(x) y) = \gamma(x) f(x)$$

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$$\boxed{\begin{aligned} y(x) &= \frac{1}{\gamma(x)} \int \gamma(x) f(x)\,dx \\ \gamma(x) &= e^{\int p(x)\,dx} \end{aligned}}$$

Existence & Uniqueness for Linear:

If $f(x)$ & $p(x)$ are continuous on (a,b) then solution exists and is unique on (a,b).

Example:

$$y' + 4y = e^{-x} \quad , y(0) = 4/3.$$

$$p(x) = 4$$

$$f(x) = e^{-x}$$

$$\gamma(x) = e^{\int 4dx} = e^{4x}$$

$$e^{4x} y' + e^{4x} 4y = e^{4x} e^{-x} = e^{3x}$$

$$\frac{d}{dx}(e^{4x} \cdot y) = e^{3x}$$

$$e^{4x} \cdot y = \int e^{3x}\,dx = \frac{1}{3} e^{3x} + c$$

c is useful for initial condition.

$$y = e^{-4x}\left(\frac{1}{3} e^{3x} + c\right)$$

$$= \frac{1}{3} e^{-x} + ce^{-4x}$$

at $x = 0$, $y = 4/3$

$$y = \frac{4}{3} = \frac{1}{3} + c \Rightarrow c = 1.$$

$$y = \frac{1}{3} e^{-x} + e^{-4x}.$$

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Bernoulli Equation:

$$y' + p(x)y = q(x) \cdot y^n$$

we will do change of variables

$$u = y^{1-n}$$

This transforms into linear first order.

$$u' = (1-n)y^{-n}y'$$

$$y' = \frac{u' y^n}{(1-n)} \quad ; \qquad y^{-n}y' = u'(1-n)$$

$$y^{-n}y' + p(x)\, y^{1-n} = q(x)$$

$$\frac{1}{(1-n)} u' + p(x)\, u = q(x)$$

This is linear first order differential eq^n.

Ex: $y' - 5y = -\frac{5}{2} x y^3$

$$u = y^{-2}$$

$$u' = \cancel{y}\ -2y^{-3}y'$$

$$y^{-3}y' - 5y^{-2} = -\frac{5}{2}x$$

$$-\frac{u'}{2} - 5u = \frac{-5}{2}x$$

$$u' + 10u = 5x$$

$$p(x) = 10$$

$$r(x) = e^{\int p(x)\,dx} = e^{10x}$$

$$e^{10x} \cdot u' + 10 e^{10x} u = e^{10x} \cdot 5x$$

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$$\frac{d}{dx}\left(e^{10x}\cdot u\right) = e^{10x}\cdot 5x$$

$$e^{10x}\cdot u = \int e^{10x}\cdot 5x \, dx$$

$$e^{10x}\cdot u = \frac{1}{10}e^{10x}\cdot 5x - \frac{1}{10}\int e^{10x}\cdot 5 \, dx$$

$$e^{10x}\cdot u = \frac{e^{10x}}{2}\cdot x - \frac{1}{100}\cdot 5\cdot e^{10x} + c$$

$$u = \frac{x}{2} - \frac{1}{20} + c\,e^{-10x}$$

$$u = y^{-2}$$

$$\frac{1}{y^2} = \frac{x}{2} - \frac{1}{20} + ce^{-10x}$$

Autonomous Equations

$$\frac{dy}{dt} = f(t,y) \rightarrow \text{general d.e.}$$

independent dependent

$$\frac{dy}{dt} = f(y)$$

dependent

An autonomous ODE only depends on the dependent variable.

ex: $$\frac{dy}{dt} = (1+y)(1-y)$$

$$\frac{dy}{dt} = 0 \rightarrow y=-1 \text{ and } y=1 \text{ (critical points)}$$

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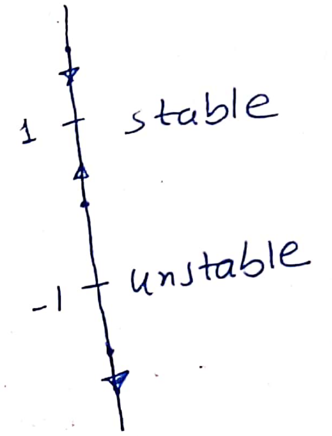
$\frac{dy}{dt} = f(y)$

Values where $f(y)=0$ are equilibrium solutions.

An equilibrium solution $y=a$ is asymptotically stable if solutions that start near a tend towards a as~~t~~ $t \to \infty$.

An equilibrium solution $y=a$ is unstable if solutions that start near a, leave a as $t \to \infty$.

ex: $\frac{dy}{dt} = (1+y)(1-y)$



$f(2) = (1+2)(1-2) = -3.$ ↓ (tend towards 1)

$f(0) = (1+0)(1-0) = 1$ ↑ (tend towards 1)

Above & below equilibrium points, the slope tends toward equilibrium point.

(Negative above 1 & positive below 1)

$f(-2) = (-1)(3) = -3.$ ↓

-1 is an unstable equilibrium point.

The solution that starts near there tends away.

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Logistic Growth DE

$y' = ay \rightarrow y(t) = ~~a~~ y_0 e^{at}$.

Let y denote the proportion of a maximal population.

$$y' = ay(1-y)$$

y is the proportion of max population, it is b/w 0 & 1.

Equilibrium solutions:

$$ay(1-y) = 0$$

$$y = 0 \quad \& \quad y = 1.$$

$y=0 \downarrow$ No population.

$y=1 \searrow$ No growth.

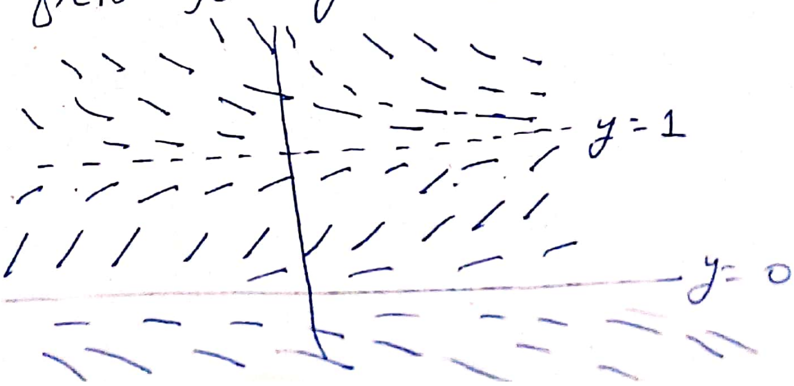
Now, let y denote total population with carrying capacity. K.

$$y' = ay\left(1 - \frac{y}{K}\right)$$

Equilibrium solution:

$$y = 0 \ \& \ y = K.$$

slope field for $y' = ay(1-y)$.



$y' = ay(1-y) \quad y(0) = y_0$

Any autonomous eqn is a separable eqn.

$$\int \frac{1}{y(1-y)}\, dy = \int a\, dt$$

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = at + c$$

$$(\ln|y| - \ln|1-y|) = at + c$$

$$\ln\left|\frac{y}{1-y}\right| = at + c$$

$$\frac{y}{1-y} = e^{at} \cdot c$$

at $t=0$; $c = \dfrac{y_0}{1-y_0}$

$$\frac{y}{1-y} = e^{at} \frac{y_0}{1-y_0}$$

$$y(t) = \frac{1}{1+\left(\frac{1}{y_0} - 1\right)e^{-at}}$$

$$\lim_{t \to \infty} y(t) = 1.$$

2nd order Linear ODE:

$$y'' + p(x)y' + q(x)y = f(x)$$

$f(x) = 0 \Rightarrow$ Homogeneous.

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Theorem: Existence & Uniqueness

For the IVP

$$y'' + p(x)y' + q(x)y = f(x)$$

$$y(a) = b_0, \quad y'(a) = b_1$$

If p, q, f continuous on interval I about a, then there exists a unique solution on I.

Theorem: Superposition

Suppose y_1 & y_2 solve

$$y'' + p(x)y' + q(x)y = 0$$

Then $c_1 y_1 + c_2 y_2 = y$ also solves.

Defn: y_1 & y_2 are linearly independent on an interval if they are not a scalar multiple of each other.

ex: $\sin x$ & $\cos x$ are linearly independent.

ex: x^2 & $2x^2$ are linearly dependent.

Theorem: Suppose there are two linearly independent solutions y_1 & y_2 to

$$y'' + p(x)y' + q(x)y = 0$$

then the general solution is

$$y = c_1 y_1 + c_2 y_2.$$

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ex: $y'' + y = 0$

$y_1 = \sin x \quad y_2 = \cos x$

$y = c_1 \sin x + c_2 \cos x$ is the general solⁿ

Note:

We don't have a general method in 2^{nd} order like integrating factor method for 1^{st} order.

Constant Coefficient ODEs

$$y'' - y' - 6y = 0$$

All coefficients are constant.

Guess: $y = e^{rt}$

$$r^2 e^{rt} - re^{rt} - 6e^{rt} = 0$$

Characteristic eqⁿ:

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r_1 = 3 \quad r_2 = -2$$

General solⁿ: $y = c_1 e^{3t} + c_2 e^{-2t}$.

e^{3t} and e^{-2t} are linearly independent.

We need 2 different initial conditions to find c_1 & c_2.

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$y(0)=1, \; y'(0)=2$

$y = c_1 e^{3t} + c_2 e^{-2t}$

$y' = 3c_1 e^{3t} - 2c_2 e^{-2t}$

$y(0) = 1 = c_1 + c_2$

$y'(0) = 2 = 3c_1 - 2c_2$

$c_1 = 4/5 \quad c_2 = 1/5$

$y = \frac{4}{5} e^{3t} + \frac{1}{5} e^{-2t}.$

Constant coefficient ODEs :

$ay'' + by' + cy = 0$

Guess : $y = e^{rt}$

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

Characteristic eq^n:

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1) $b^2 - 4ac > 0$; two real, distinct roots

2) $b^2 - 4ac = 0$; One repeated real root.

3) $b^2 - 4ac < 0$; Complex pair of roots.

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1) $b^2-4ac>0$

Two real, distinct root

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2) $b^2-4ac=0$

One repeated real root

General solution:

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

This works as

a) te^{rt} is a solution

b) te^{rt} linearly independent from e^{rt}.

3) $b^2-4ac<0$

Complex pair of roots

$$r = \alpha \pm i\beta.$$

$$y = c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$

$$y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t} \cdot e^{i\beta t}$$

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$y_2 = e^{\alpha t - i\beta t} = e^{\alpha t} \cdot e^{-i\beta t}$$

$$= e^{\alpha t}(\cos\beta t - i\sin\beta t)$$

$$\frac{y_1+y_2}{2} = e^{\alpha t}\cos\beta t \qquad \frac{y_1-y_2}{2i} = e^{\alpha t}\sin\beta t$$

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$y = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t.$

Example:

$$y''' + y' = 0$$

Order : 3

Constant coefficient DE.

Guess : $y = e^{rt}$

Characteristic eqn

$$r^3 + r = 0$$

$$r(r^2+1) = 0 \Rightarrow r(r+i)(r-i) = 0$$

$$r = 0, i, -i$$

$$y = c_1 e^{0t} + c_2 e^{it} + c_3 e^{-it}$$

$$= c_1 + c_2 \cos t + c_2 \sin t + c_3 \cos t - c_3 \sin t$$

$$= c_1 + c_4 \cos t + c_5 \sin t.$$

Example

$$y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$$

Guess $y = e^{rt}$

$$\Rightarrow r^4 - 3r^3 + 3r^2 - r = 0$$

$$r(r^3 - 3r^2 + 3r - 1) = 0$$

$$r(r-1)^3 = 0$$

$$r = 0, 1, 1, 1.$$

$$y = c_1 e^{0t} + c_2 e^t + c_3 t e^t + c_4 t^2 e^t.$$

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Linear Independence of Functions & Wronskian

Defⁿ: y_1 & y_2 are linearly independent on an interval if they are not a scalar multiple of each other.

ex: $\sin x$ & $\cos x$ are linearly independent.

Defⁿ: $y_1 \cdots y_n$ are linearly independent on an interval if $a_1y_1 + \cdots + a_ny_n = 0$ implies $a_1 = \cdots = a_n = 0$.

ex: $e^x, e^{-x}, \cosh x = \frac{e^x+e^{-x}}{2}$.

$$\frac{1}{2}e^x + \frac{1}{2}e^{-x} - 1\cosh x = 0$$

So, linearly dependent.

The Wronskian:

$$w(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Suppose y_1 and $y_2 = ay_1$ are linearly dependent

$$w(t) = \begin{vmatrix} y_1 & ay_1 \\ y_1' & ay_1' \end{vmatrix}$$

$$= y_1 ay_1' - ay_1y_1' = 0.$$

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For 'nice' functions $y_1, \dots, y_n$ they are linearly independent iff $w(t) \neq 0$ at some point t.

Nice - analytic functions where there is Convergent series like sin x, cos x, e^x

In context of DE if y_1 & y_2 are solutions, then they are nice.

Are e^t, te^t are linearly independent?

$$w(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^t(e^t + te^t) - te^t e^t$$

$$= e^{2t} \neq 0.$$

Higher order Differential Equations

$$y^{(4)} + x^2 y^{(3)} + e^x y = 3$$

General:

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_0(x) y = f(x)$$

Theorem: Existence & Uniqueness

For the IVP,

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_0(x) y = f(x)$$

$$y(a) = b_0, \; y'(a) = b_1 \; \dots \; y^{(n-1)}(a) = b_{n-1}.$$

$\rightarrow$

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If all p_i & f are continuous on interval I about a. Then there exists a unique solution on I.

Theorem: Superposition

Suppose $y_1 \cdots y_n$ solve

$$y^n + P_{n-1}(x) y^{(n-1)} + \cdots + P_0(x) y = 0$$

Then $y = c_1 y_1 + \cdots + c_n y_n$ solves.

Theorem: General solution

Suppose $y_1 \cdots y_n$ solve

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_0(x) y = 0,$$

Then $y = c_1 y_1 + \cdots + c_n y_n$ is the general solution iff $W(t) \neq 0$ for some t_0.

Method:

1. For n^{th} order, find n solutions $y_1 \cdots y_n$
2. Check that $W(t) \neq 0$ at some point.
3. General solution: $y = c_1 y_1 + \cdots + c_n y_n$.

Constant coefficient ODE example.

$$y'' - y' - 6y = 0$$

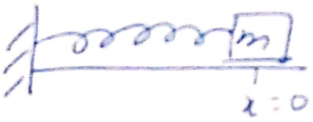
$$y_1 = e^{3t}, \; y_2 = e^{-2t}$$

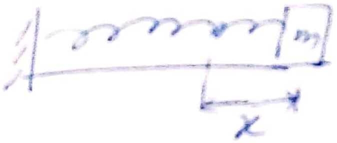
$$W = \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t} \end{vmatrix} = -2e^{t} - 3e^{t} = -5e^{t} \neq 0$$

$$y = c_1 e^{3t} + c_2 e^{-2t}$$

Mechanical Vibrations & Hooke's Law

Assume frictionless surface & we move the mass by a displacement x.





Hooke's Law: $F = -kx$

k → spring Constant

x → Displacement from equilibrium.

$$mx'' = -kx$$

$$mx'' + kx = 0$$

Guess: $x = e^{rt}$

$$mr^2 - K = 0$$

$$r = \pm i\sqrt{K/m} = \pm i\omega_0.$$

General sol^n:

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$x \overset{?}{=} c\cos(\omega_0 t - r)$$

$$= c(\cos(\omega_0 t)\cos r + \sin(\omega_0 t)\sin r).$$

$$A = c\cos r \qquad B = c\sin r$$

$$c = \sqrt{A^2+B^2}; \quad \tan r = B/A$$

$x = c\cos(\omega_0 t - r)$. is the sol^n.

Now, we take friction into account.

Friction force $f_f = -cx'$.

Friction force depends on velocity.

Overall eq^n:

$$mx'' + cx' + kx = 0$$

Guess : $x = e^{rt}$

characteristic eq^n:

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

1. $c^2 - 4mk < 0$ (Underdamped)
   Complex pair

2. $c^2 - 4mk > 0$ (overdamped)
   Two real distinct roots.

3. $c^2 - 4mk = 0$ (Critically damped)
   Real repeated roots.

Underdamped

1. $c^2 - 4mk < 0$.

$$r = \frac{-c}{2m} \pm \underbrace{\frac{\sqrt{c^2 - 4mk}}{2m}}_{i\omega_1}$$

$$x = e^{-c/2m\, t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right)$$

$$= e^{-c/2m\, t}\left(C\cos(\omega_1 t - \gamma)\right).$$

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Overdamped

2. $c^2 - 4mk > 0$

No oscillations.

$$x = Ae^{r_1 t} + Be^{r_2 t}$$

r_1 & r_2 are both negative.

Critically damped

3. $c^2 - 4mk = 0$

$$x = Ae^{rt} + Bte^{rt}.$$

r is ~~r~~ negative.

t in second term increases.

Exponential term try to drag it down to 0 but t increases it.

Undetermined coefficients

$$y'' - 2y' - 3y = 3e^{2t}.$$

Particular solution y_p solves non-homogeneous

y_h solves homogeneous : $y'' - 2y' - 3y = 0$.

$y_h + y_p$ is the solⁿ of eqⁿ.

And $y_p - \tilde{y}_p$ solves homogeneous solution.

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$$y'' - 2y' - 3y = 3e^{2t}$$

Step 1: Solve the homogeneous

Try e^{rt}: $r^2 - 2r - 3 = 0$

$$r_1 = 3, \quad r_2 = -1$$

$$y_h = c_1 e^{3t} + c_2 e^{-t}.$$

Step 2: Find a solution to non-homogeneous

Guess $y_p = Ae^{2t}$.

$$A4e^{2t} - A4e^{2t} - 3Ae^{2t} = 3e^{2t}$$

$$-3A = 3$$

$$A = -1.$$

$$y_p = -e^{2t}.$$

Step 3: General sol^n is

$$\underset{\underset{\text{General}}{\downarrow}}{y_g} = \underset{\underset{\text{homogeneous}}{\downarrow}}{y_h} + y_p \rightarrow \text{particular.}$$

$$y_g = c_1 e^{3t} + c_2 e^{-t} - e^{2t}.$$

Get constants from initial conditions.

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Ex 2: $y'' - 2y' - 3y = 3e^{-t}$

$$y_h = c_1 e^{3t} + c_2 e^{-t}$$

e^{-t} is also present for homogeneous solution

Problem: Ae^{-t} can't be a particular solution.

Try $y_p = Ate^{-t}$

$$(Ate^{-t})'' - 2(Ate^{-t})' - 3(Ate^{-t}) = 3e^{-t}$$

$$(-2Ae^{-t} + A\cancel{te^{-t}}) - 2(Ae^{-t} - A\cancel{te^{-t}}) - 3A\cancel{te^{-t}} = 3e^{-t}.$$

$$-4Ae^{-t} = 3e^{-t}$$

$$A = -3/4.$$

General sol^n:

$$y_g = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} te^{-t}.$$

Non-homogeneity	Guess
e^{rt}	Ae^{rt}
$\sin(rt)$ or $\cos(rt)$	$A\sin(rt) + B\cos(rt)$
Degree n polynomial	$A_0 + A_1 t + \dots + A_n t^n$

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- Multiply by t^s as need to avoid matching homogeneous.
- Add / multiply different types together.

ex: $y'' - 2y' - 3y = t^2 + 3e^{-t} \cos(4t)$

$y_h = c_1 e^{3t} + c_2 e^{-t}$

$y_p = (A + Bt + Ct^2) + D(e^{-t} \cos(4t))$
$+ E(e^{-t} \sin(4t))$.

Variation of parameters

$$y'' + y = \tan(x)$$
$$\downarrow$$
Non-homogeneous

$y'' + y = 0$ homogeneous

$y_h(x) = c_1 \sin(x) + c_2 \cos(x)$.

Let $y(x) = u_1(x) \sin x + u_2(x) \cos(x)$

& we guess that c_1 & c_2 are some function of x

$y = u_1 \sin x + u_2 \cos x$

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$y' = \underline{u_1' \sin x} + u_1 \cos x + \underline{u_2' \cos x} - u_2 \sin x$

Let us impose a constraint:

$$u_1' \sin x + u_2' \cos x = 0.$$

So, $y' = u_1 \cos x \cancel{+} - u_2 \sin x$

$y'' = u_1' \cos x - u_1 \sin x - u_2' \sin x - u_2 \cos x$

Now, we put into original eq^n.

$$y'' + y = \tan x$$

$$u_1' \cos x \cancel{+} - u_1 \cancel{\sin x} - u_2' \sin x - u_2 \cancel{\cos x} + u_1 \cancel{\sin x} + u_2 \cancel{\cos x} = \tan x.$$

$$u_1' \cos x - u_2' \sin x = \tan x \quad -①$$

We also have -

$$u_1' \sin x + u_2' \cos x = 0. \quad -②$$

$$u_1' \cos^2 x - u_2' \sin x \cos x = \tan x \cos x \quad -③$$

$$u_1' \sin^2 x + u_2' \sin x \cos x = 0 \quad -④$$

Add ③ & ④.

$$u_1' (\cos^2 x + \sin^2 x) = \tan x \cos x.$$

$$u_1' = \sin x$$

So, $u_2' = \tan x \sin x.$

$u_1 = \int \sin x \, dx = -\cos x + C_1$

$u_2 = \int \tan x \sin x \, dx = \frac{1}{2} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| + \sin x + C_2$

So, solution is

$y = u_1 \sin x + u_2 \cos x.$

This type of solution works for:

$$\underbrace{y'' + p(x) y' + q(x) y}_{\text{Linear}} = g(x)$$

- The homogeneous eqⁿ has two linearly independent solutions y_1 & y_2

- Guess $y = u_1 y_1 + u_2 y_2$

- Solves as $u_1 = -\int \frac{y_2 g}{y_1 y_2' - y_2 y_1'} dx$

$$u_2 = -\int \frac{y_1 g}{y_1 y_2' - y_2 y_1'} dx.$$

The denominator is Wronskian. Now, it will not be 0 if y_1 & y_2 are linearly independent.

Variation of parameters almost always work. depending on our ability ~~of~~ to integral.

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Solve ODEs using infinite series

$y' = y \Rightarrow y = Ce^x$.

Suppose $y = \sum_{n=0}^{\infty} C_n x^n$ on I.

Recap:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n}$$

$y' = y$

Suppose $y = \sum_{n=0}^{\infty} C_n x^n$ on I

$y' = \sum_{n=0}^{\infty} C_n n x^{n-1}$ on I

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$$\cdot\, y' = \sum_{n=0}^{\infty} C_n\, n\, x^{n-1} \quad \text{on } I$$

$$= \sum_{n=1}^{\infty} C_n\, n\, x^{n-1}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} C_n\, n\, x^{n-1} = \sum_{n=0}^{\infty} C_n x^n ,$$

Shift : $n \to n+1$.

$$\sum_{n=0}^{\infty} C_{n+1}\,(n+1)\, x^n = \sum_{n=0}^{\infty} C_n x^n .$$

$$\cancel{C_n(}\; C_{n+1}\,(n+1)) = C_n .$$

$$C_{n+1} = \frac{C_n}{(n+1)} \qquad \text{(Recurrence relation)}$$

$C_0 =$ Some constant

$$C_1 = \frac{C_0}{1}$$

$$C_2 = \frac{C_1}{2} = \frac{C_0}{1\cdot 2}$$

$$C_3 = \frac{C_2}{3} = \frac{C_1}{2\cdot 3} = \frac{C_0}{1\cdot 2\cdot 3}$$

$$C_n = \frac{C_0}{n!}$$

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = C_0 e^x .$$

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Convergen Tests:

Ratio test:

1.) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

3) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \Rightarrow$ Inconclusive.

example

$$y = C_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{C_0 x^{n+1}}{(n+1)!}}{\frac{C_0 \cdot x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x}{(n+1)} \right| = 0.$$

By ratio test it converges for all values of x.

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2^{nd} order, linear, homogeneous

analytic coefficients

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$$

$f(x)$ is analytic at a if it has a convergent series on an open interval about a.

We divide by $A(x)$.

$$y'' + \underbrace{p(x)}_{B/A}y' + \underbrace{Q(x)}_{C/A}y = 0.$$

$x = a$ is an ordinary point if $P(x)$ & $Q(x)$ are analytic at $x = a$.

Otherwise singular point.

Theorem: For an ordinary point a,

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$$

has two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} C_n (x-a)^n.$$

The radius of convergence is at least as large as distance to nearest singular point.

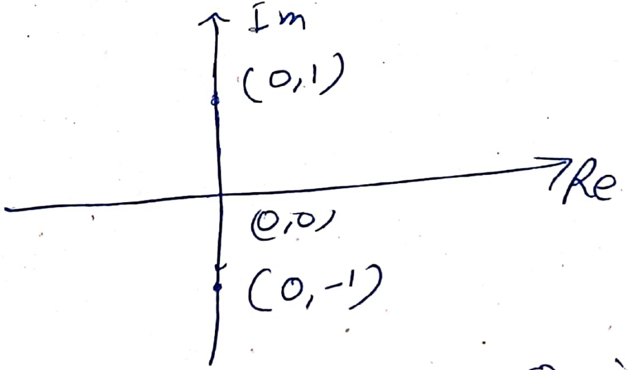
Ex: $(1+x^2)y'' - y' + y = 0$

$P(x)$, $Q(x)$ analytic everywhere
but $x = \pm i$.

Expanding about $a=0$.

$$y(x) = \sum_{n=0}^{\infty} C_n x^n.$$

$x = \pm i$, on complex plane we can represent as



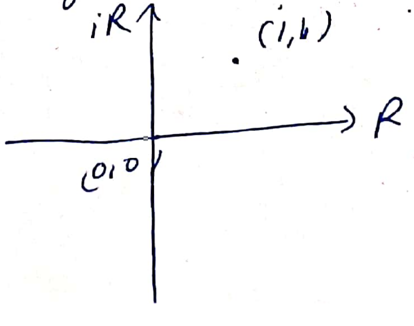
So, distance from 0 is 1, so Radius of convergence is 1.

So, solution ~~convee~~ converges within that distance.

~~Consider 2nd order, linear homogeneous~~

~~Analytic coefficients~~

If we have a DE that is analytic everywhere but at $x = 1+i$.



Radius of convergence $= \sqrt{1^2+1^2} = \sqrt{2}$.

Series Solutions

ex: $y'' - xy = 0$

Assume $y = \sum_{n=0}^{\infty} C_n x^n$

$$y'' = \sum_{n=0}^{\infty} C_n (n)(n-1) x^{n-2}$$

$$\sum_{n=0}^{\infty} C_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} C_n x^{n+1} = 0.$$

$\downarrow$

$n = n+3$

$\cancel{\Sigma}$

$$\sum_{n=-3}^{\infty} C_{n+3} (n+3)(n+2) x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1}.$$

For $n=-3$ & $n=-2$, $(n+3)(n+2)$ becomes 0.

$$\Rightarrow \sum_{n=-1}^{\infty} C_{n+3} (n+3)(n+2) x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1}.$$

$$\Rightarrow C_2 \cdot 2 \cdot 1 \cdot x^0 + \sum_{n=0}^{\infty} C_{n+3} (n+3)(n+2) x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1}.$$

$\Rightarrow C_2 = 0.$ if we put $n=0$.

$$\Rightarrow \sum_{n=0}^{\infty} C_{n+3} (n+3)(n+2) x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$C_{n+3} (n+3)(n+2) = C_n$$

We have coefficient $C_0, C_1, \& C_2$

we know $C_2 = 0$.

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$C_{n+3}(n+3)(n+2) = C_n$

C_0

C_1

$C_2 = 0$

$C_3 = \frac{C_0}{3 \cdot 2}$

$C_4 = \frac{C_1}{4 \cdot 3}$

$C_5 = \frac{C_2}{5 \cdot 4} = 0.$

$C_6 = \frac{C_3}{5 \cdot 6} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6}$

$C_7 = \frac{C_4}{7 \cdot 6} = \frac{C_1}{7 \cdot 6 \cdot 4 \cdot 3}$

$C_8 = 0.$

$\Rightarrow C_{3n} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}$

$\Rightarrow$ ~~C_n~~ $C_{3n+1} = \frac{C_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}$

& $C_{3n+2} = 0$

$$y = C_0\left(1 \cdot x^0 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}\right).$$

$$+ C_1\left(1 \cdot x^1 + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}\right]$$

Conclusion

1^{st} order

- Separable ? → Separation of variables $y' = x/y$
- Linear ? → Integrating Factors $y' + 4y = e^{-x}$.
- Bernoulli? → Substituting to Linear

  $y' + p(x)y = q(x)y^n$.
- Autonomous → Equilibrium Analysis $y' = f(y)$

2^{nd} Order

- Constant coefficient → Guess e^{rt}

  $y'' - 2y' - 3y = 0$
- Non-homogeneous → Undetermined coefficients or variation of parameters

  $y'' - 2y' - 3y = 3e^{2t}$.
- Linear → Series Solution $y' - xy = 0$.

Discontinuities /Spikes → Laplace Transform.

1^{st} order : $y' + x^2 y = e^x$

2^{nd} order : $y'' + xy' + x^2 y = e^x$.

Linear : $y''+xy'+x^2y = e^x$

Non-linear: $y''\cdot e^{y'} = x/y$

Homogeneous Linear: $y''+p(x)y'+q(x)y=0.$

1st order: Separable

$$y'y = x$$

$$\int y\,dy = \int x\,dx + c$$

$$\frac{y^2}{2} = \frac{x^2}{2}+c.$$

1st order Linear:

$$y'+p(x)y = r(x)$$

$$y'+4y = e^{-x}$$

Integrating factor

$$u = e^{\int 4dx} = e^{4x}$$

$$e^{4x}\cdot y' + e^{4x}\cdot 4y = e^{4x}\cdot e^{-x}$$

$$\frac{d}{dx}(e^{4x}\cdot y) = e^{3x}$$

$$e^{4x}\cdot y = \frac{e^{3x}}{3} + c \Rightarrow y = \frac{e^{-x}}{3} + ce^{-4x}.$$

1^{st} order substitutions:

- Bernoulli: $y' + p(x) \cdot y = q(x) y^n$

  $v = y^{1-n}$ converts to linear!

- Homogeneous: $y' = F(y/x)$

  $v = y/x$ converts to separable.

1^{st} order Autonomous:

$$y' = f(y)$$

We can draw slope fields. And easier to solve.

We look for stable & unstable points.

2^{nd} order constant coefficient & Homogeneous:

$$y'' - 2y' - 3y = 0$$

Guess $y = e^{rt}$

$r^2 - 2r - 3 = 0$ Characteristic eq^n

$$y = c_1 e^{3t} + c_2 e^{-t}$$

For $ay'' + by' + cy = 0$

Roots will be →

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. $b^2 - 4ac > 0$

Two real, distinct roots

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

2. $b^2 - 4ac = 0$

One repeated real root

$$y = c_1 e^{rt} + ~~c_2e~~ c_2 t e^{rt}$$

3. $b^2 - 4ac < 0$

complex pair of roots

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

2nd order Non homogeneous :

$$y'' - 2y' - 3y = \underline{3e^{2t}}$$

1. Solve homogeneous equation in general y_h.
2. Find one solution to full equation y_p.
3. Add solutions together $y_h + y_p$.

Undetermined coefficient.

Guess $y_p = Ae^{2t}$.

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Non-homogenity	Guess
e^{rt}	$A e^{rt}$
$\sin(rt)$ or $\cos(rt)$	$A \sin(rt) + B \cos(rt)$
Degree n polynomial	$A_0 + A_1 t + \dots + A_n t^n$

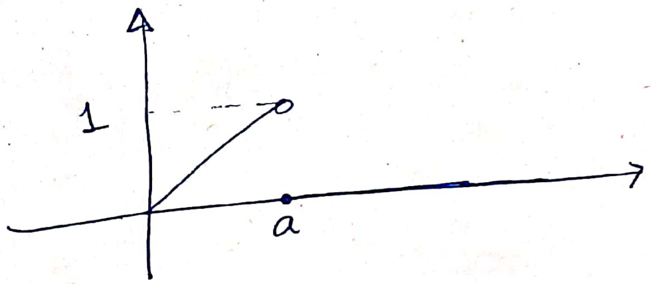
- Multiply by ts as need to avoid matching homogeneous.
- Add/Multiply different types together.

For any other type of non-homogenity we use variation of parameters.

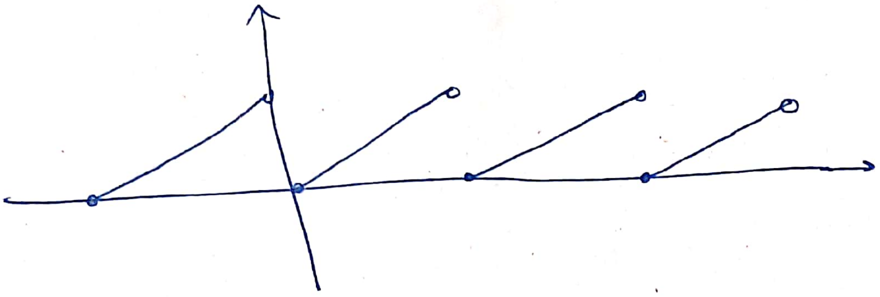
Laplace Transform :

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

$$r(x) = \begin{cases} x, & x \in [0, a) \\ 0, & x \notin [0, a) \end{cases}$$



$r(x) = x$ on $[0,1)$ with period 1.



$r(x) = \delta_a$.

Best for $r(x)$...

- Discontinuous
- Periodic
- Infinite Spike.

Use Laplace Transform

$$\begin{tikzcd} \text{Differential } eq^n \arrow[r, "\mathcal{L}"] & \text{Algebraic } eq^n \arrow[d] \\ \text{Solution.} & \text{Solve} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

Series solution

$$y'' - xy = 0$$

Given $y = \sum_{n=0}^{\infty} C_n x^n$.

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