L1 (Intro)

Vector Field - At any point we have a dir^n & how fast a ~~parameter is moving~~ that point is moving. So if we plot for all points we get a vector field.

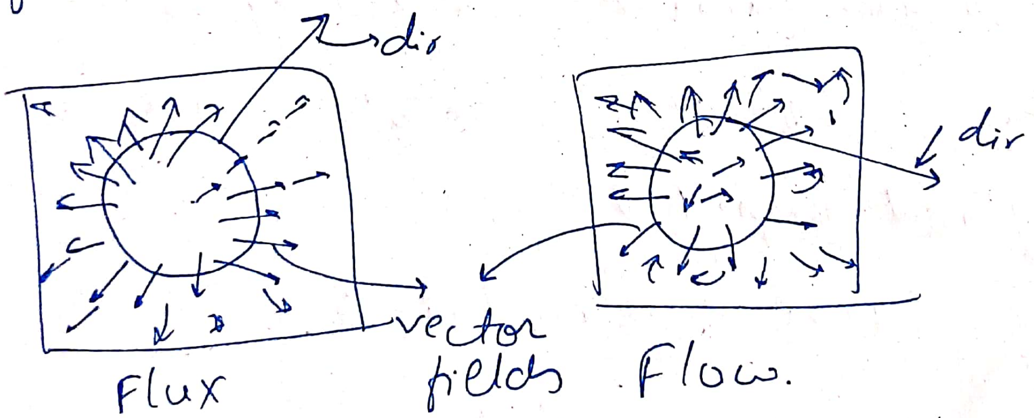
Curve - A path in space. As time goes on we move along a path.

We can also have surfaces. Like closed surface is a sphere. Or an open surface with a boundary.

we will see line integral, surface integral,

we are interested in curves, surfaces & vector fields.

If we have a curve in a vector field.



Flux Flow.

Normal to Curve (Flux)

Tangential to curve. (Flow)

Flux - To what degree is the vector field moving ~~out~~ across the boundary.
ie To what degree is normal of curve aligned to vector field.

Flow - To what degree is the vector field aligned to the tangent of the curve.

Big question: How local properties relate to global properties?

Local properties: Something that happen that every single point. (Different value of property at different points)

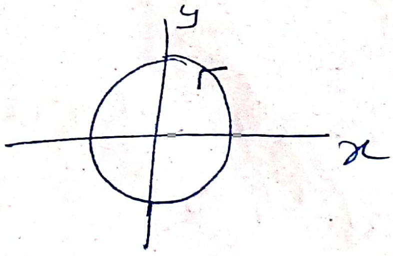
Global property: Overall property of the curve. (Like tendency of particle to flow around a boundary curve).

(Green, Stokes theorem relate local to global properties)

L2 Curve, Parameterization.

unit circle $x^2+y^2=1$

$\vec{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} \quad 0 \le t \le 2\pi$



$\vec{r}(t) = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} \qquad 0 \le t \le \pi.$

(Twice as fast than last)

For a 3D vector-

$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}.$

To get a <u>tangent vector</u>

$r'(t) = g'(t)\hat{i} + h'(t)\hat{j} + k'(t)\hat{k}$

$= v'(t)$ (velocity vector)

Unit Tangent $\vec{T} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

$\vec{T}$ = Tangent vector (unit)

* The distance along the curve is given by arclength

$$\boxed{\text{Arclength} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + k'(t)^2}\, dt}$$

$$= \int_a^b |\vec{v}(t)|\, dt$$

(velocity x ~~distance~~ time)

~~So~~, our parameterization does not matter for getting arclength.
(we can have different type of parameter depending on convienence)

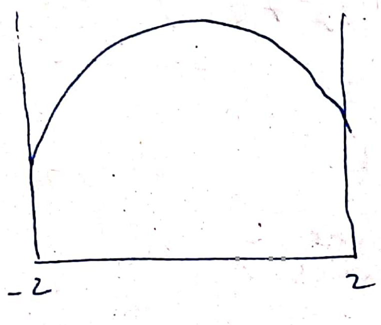
Arclength Parameter

$$\boxed{s(t) = \int_a^t |\vec{v}(\tau)|\, dt}$$

So, we have arclength parameter (s) & time parameter (t).

Arclength Parameter (s)	Time parameter (t)
• Intrinsic to geometry of curve	• Intrinsic to object along curve
• One choice	• Many choices
• Hard to compute	• Easy to compute

L3 Line Integral (Path integral)

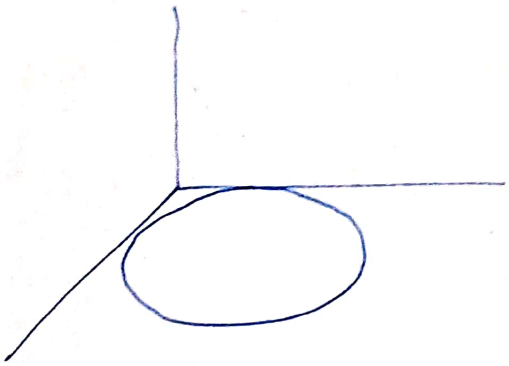


$f(x)$ = height above line segment i.e. domain (here -2, 2)

Area under curve

$$= \int_a^b f(x)\, dx$$

Now in 3D., our domain is in 2D.

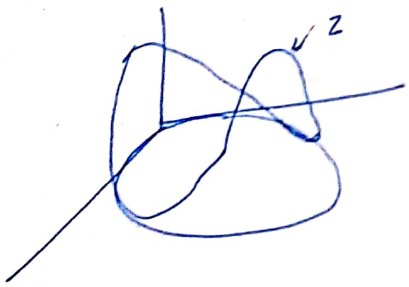


for the curve in 2D

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$$
$$t \in [a,b]$$

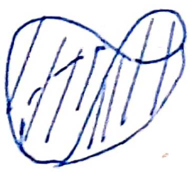
By paramaterization, we can give single input t to get a 2D curve.

Now, let $z = f(x,y)$ i.e now instead of $z=0$ we have some value for z.



$$z = f(x,y)$$
$$z = f(g(t), h(t))$$

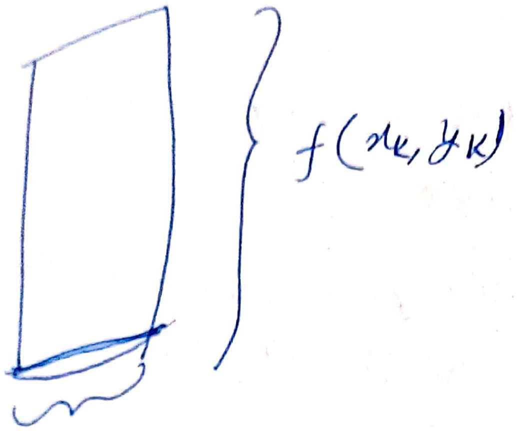
Now, the question is what is the area of the surface above the curve.



Now, we break into small pieces. ~~to~~ (rectangular chunks.

To get area:

1) Break $[a, b]$ into n segments.
2) Approximate curve with n segments
3) Approximate area with n rectangles.



Δs_k → delta arc length.

$$\Delta A_k = f(x_k, y_k)\, \Delta s_k .$$

<u>Line integral</u> over a curve:

$\int_C f(x,y)\, ds$ → infinitesimal increase in arc length s.

$$= \sum_{k=1}^{\infty} f(x_k, y_k)\, \Delta s_k$$

$$= \boxed{\int_C f(x,y)\, ds.}$$

we need a simpler formula for Δs_k.

$$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\Delta A_k = f(x_k, y_k)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Trick: divide & multiply by Δt.

$$= f(x_k, y_k)\sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2}\,\Delta t$$

$\downarrow n \to \infty$

$$\boxed{dA = f(g(t), h(t))\sqrt{(g'(t))^2 + (h'(t))^2}\,dt}$$

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(g(t), h(t))\sqrt{g'(t)^2 + h'(t)^2}\,dt}$$

(we converted arc length parameter to time parameter).

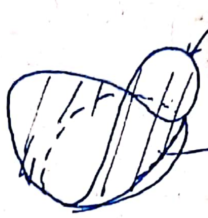
We know $\int\sqrt{g'(t)^2 + h'(t)^2}\,dt$ = arc length.

Now, if our $f(x,y)$ is say density (mass per unit length) then an integration $\int_C f(x,y)\,ds$ gives us the mass.

L4 Examples

① Line integral of $f(x,y) = \frac{x^2+y^2}{4} + \frac{xy}{2}$ above a circle of radius 2 centered at origin.

$\Rightarrow$ it looks as



we are asked to calculate surface area going down the curve.

$$\int_C f(x,y)\,ds = \int_a^b f(x,y)\sqrt{g'(t)^2 + h'(t)^2}\,dt$$

Step 1: Parameterize

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$$
$$t \in [0, 2\pi]$$

Step 2: Write $f(x,y)$ in terms of parameter

$$f = \frac{(2\cos t)^2 + (2\sin t)^2}{4} + \frac{2\cos t \times 2\sin t}{2}$$

$$= 1 + \sin 2t$$

Step 3: Plug in formula

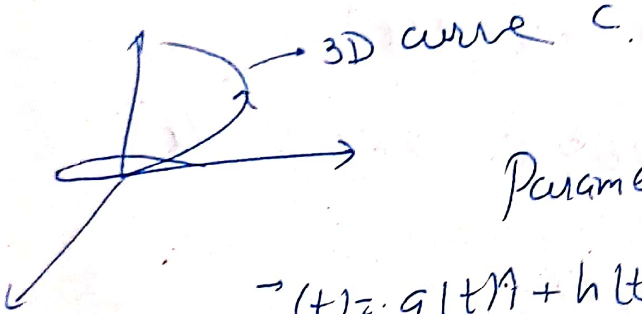
$$= \int_0^{2\pi} (1+\sin 2t)\sqrt{\cancel{-2\cos t}(-2\sin t)^2+(2\cos t)^2}\,dt$$

$$= \int_0^{2\pi} (1+\sin 2t)\times 2\,dt$$

$$= 4\pi$$

$$\int_C f(x,y)\,ds = 4\pi.$$

L5 Line integrals in 3D



Parameterize C

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$$
$$t \in [a,b]$$

For above curve

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \qquad t \in [0, 2\pi]$$

Now, we have a fn f

$$f(x,y,z) = f(g(t), h(t), k(t))$$

It is hard to draw f over the curve C.

But the basic idea of line integrals would apply.

So, now in 3D

$$\int_C f(x,y,z)\, ds = \int_a^b f(g(t), h(t), k(t)) \times \sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}\, dt$$

Example 1: A wire and $f(x,y,z)$ is linear mass density $\delta(t)$.

$$\text{Mass} = \int_a^b \delta(t) \sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}\, dt$$

density × arc length

Example 2: A pipe and $f(x,y,z)$ is cross sectional area $A(t)$

$$\text{Volume} = \int_a^b A(t) \sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}\, dt$$

↓

Area × length.

Example 3: $f(x,y,z) = 1$

$$\text{Arc length} = \int_a^b \sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}\, dt.$$

L6 Intro to vector fields

Vector field → arrow that tells what will happen at every single spot.

ex ⇒ arrow in wind speed plot.

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A vector field is a fn $\vec{F}: R^n \to \mathbb{R}^n$

$$\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$$

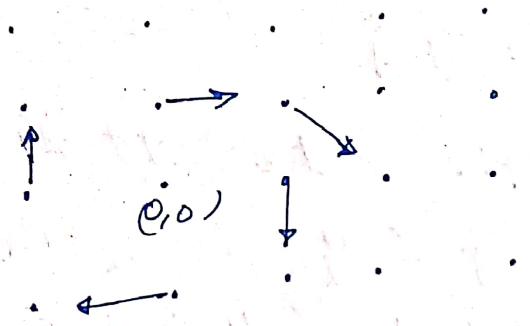
$$\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}.$$

Q Draw vector field for

$$\vec{F}(x,y) = y\hat{i} - x\hat{j}$$

$\Rightarrow$ $\vec{F}(0,0) = (0,0)$ $\quad F(1,1) = (1,-1)$

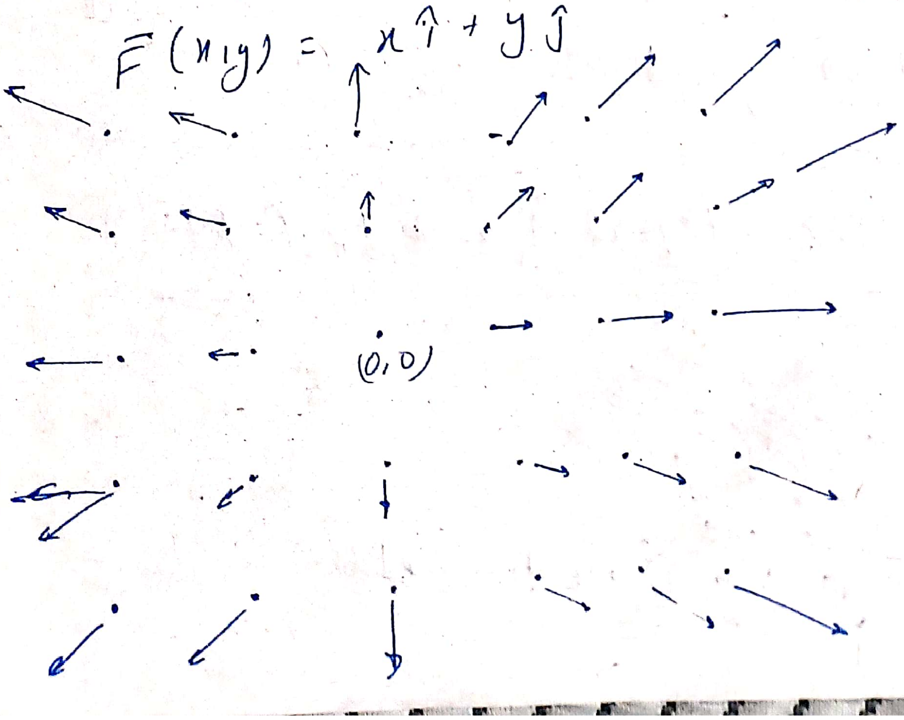
$\vec{F}(1,0) = (0,-1)$ $\quad F(0,1) = (1,0)$



Similarly we can plot for all the points & more points

Q Draw the vector field for

$$\vec{F}(x,y) = x\hat{i} + y\hat{j}$$



L7 The Gradient Field

Given a fn $\mathbb{R}^3 \rightarrow \mathbb{R}$

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

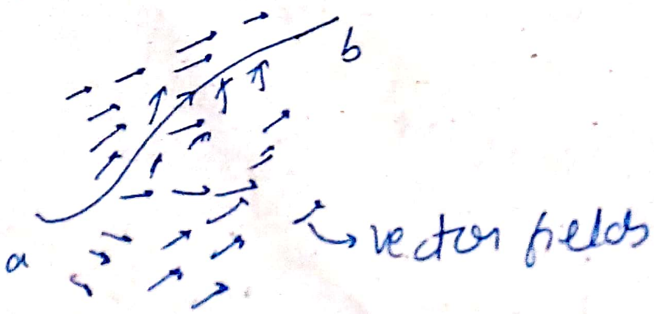
eg: $f(x,y) = x^2 + y^2$

$$\nabla f = 2x\,\hat{i} + 2y\,\hat{j}$$

we plotted the vector field $x\hat{i} + y\hat{j}$.
when we plot f & ∇f, we see that f is a paraboloid & ∇f is a vector field pointing out wards

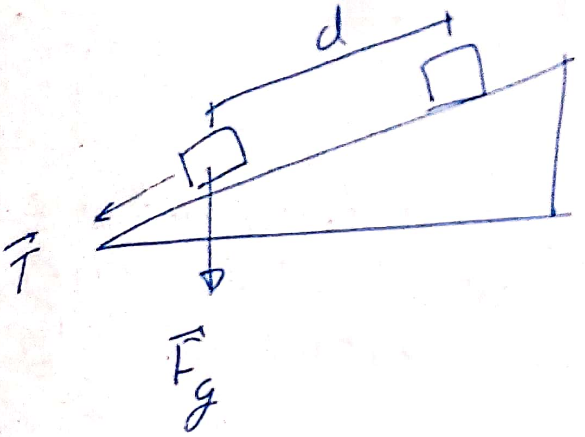
~~use~~ Gradient vector points in the direction of steepest increase of your function.

L8 Line integral of a Vector Field



The example is calculation of work.

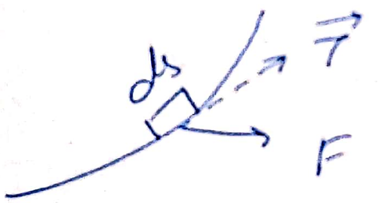
Work = Force × Distance.



$\vec{T}$ = Tangent vector $\left(\frac{\vec{v}(t)}{|\vec{v}(t)|}\right)$

$$W = \vec{F_g} \cdot \vec{T} d$$

We have a curve C & field vector $\vec{F}$. So, our work depends on tangential component of $\vec{F}$ along C. We use little unit ds



$$dW = \vec{F} \cdot \vec{T} ds$$

⇒ The line integral of $\vec{F}$ ~~ds~~ along C is

$$\int_C \vec{F} \cdot \vec{T} ds.$$

Here our integrand changed to $\vec{F} \cdot \vec{T}$.

Parameterized curve for $t \in [a,b]$

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}.$$

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$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}.$

Line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot \vec{T}\, ds$$

$\vec{T} = \dfrac{d\vec{r}}{ds}$ (how much does position vector change as we increase arc length)

$$ds = \frac{ds}{dt} \times dt$$

$$\Rightarrow \boxed{\int_a^b F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt.}$$

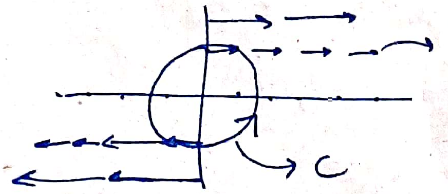
$$\int_C \vec{F} \cdot \vec{T}\, ds = \boxed{\int_a^b F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt}$$

$$= \int_a^b \vec{F} \cdot d\vec{r}.$$

L9 Example

* Compute work done by $F(x, y) = y\hat{i} + 0\hat{j}$ on particle moving along unit circle. (CCW)

$\Rightarrow$



Vector field.

$$W = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot \frac{dr}{ds}\, ds = \int_C \vec{F} \cdot \frac{dr}{dt}\, dt.$$

$$= \int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt$$

$$r(t) = (\cos t\, \hat{i}, \sin t\, \hat{j} + 0\hat{k})$$

$$F(r(t)) = \sin t\, \hat{i} + 0\hat{j}$$

$$r'(t) = -\sin t\, \hat{i} + \cos t\, \hat{j} + 0\, \hat{k}$$

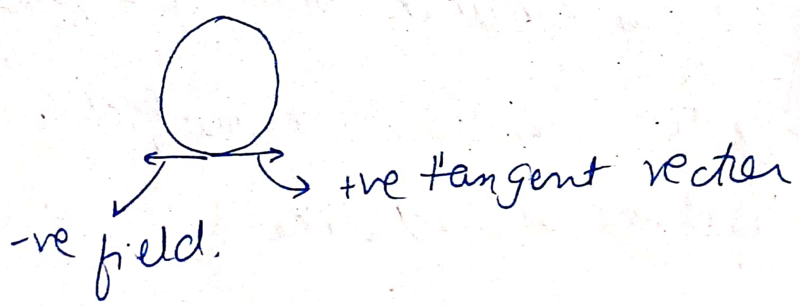
$$W = \int_0^{2\pi} (\sin t\, \hat{i} + 0\hat{j}) \cdot (-\sin t\, \hat{i} + \cos t\, \hat{j})\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt$$

$$= \int_0^{2\pi} -\frac{1}{2}(1 - \cos 2t)\, dt$$

$$= -\pi$$

We have -ve work in this context.
Our tangent vector of curve is going to left while field vector is going in right, so it is sensible that we will get -ve work.



L10 Line integral w.r.t x or y

We saw before our curve C

$r(t) = g(t)\hat{i} + h(t)\hat{j}$ ~~$+ k(t)\hat{k}$~~.

And our line integral above the curve gave us the surface area above the curve ~~till point $f(x,y)$~~.
till point $f(x,y)$

$$\int_C f(x,y)\,ds = \int_a^b f(g(t), h(t))\sqrt{g'(t)^2 + h'(t)^2}\,dt$$

Now, suppose we have the projection of curve on the x-z plane (of original curve). Now, we need to know what is the area of the projection

Line integral w.r.t. dx:

$$\int_C f(x,y)\,dx = \int_a^b f(g(t), h(t))\,g'(t)\,dt$$

This is the area we get when we project onto x-z plane.

Similarly we can calculate onto y-z plane.

Same thing we can do in vector field.

If we had vector field like

$M(x,y)\hat{i} + N(x,y)\hat{j}$

We can calculate for contribution only in x direction by making N as 0.

Field: $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

Curve : $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$

Line integral

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_a^b \left(M(x,y)\,g'(t) + N(x,y)\,h'(t)\right)dt$$

$$= \int_a^b M(x,y)\underbrace{g'(t)dt}_{dx} + \int_a^b N(x,y)\underbrace{h'(t)\,dt}_{dy}$$

$$\boxed{= \int_C M\,dx + \int_C N\,dy}$$

(Line integral)

<u>Example</u> Curve C is a portion of Parabola $y = x^2$ from (1,1) to (2,4)

compute $\int_C \frac{x}{y}\,dy$.

We need $\int \frac{x}{y}\, dy$

$\Rightarrow$ (1) Parameterize

$x = t, \quad y = t^2 \quad ; \quad t \in (1, 2)$

$r(t) = t\hat{i} + t^2\hat{j}, \quad 1 \le t \le 2$

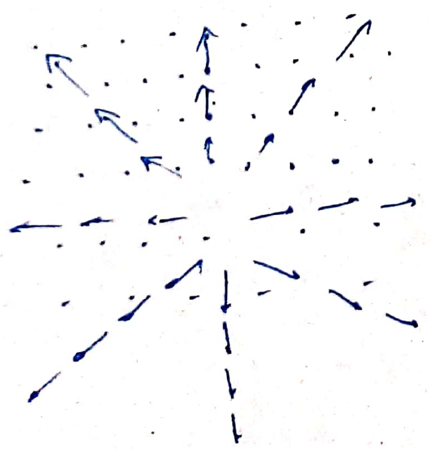
(2) Substitute into integral

$$\int_1^2 \frac{t}{t^2} \times 2t\,dt$$

$$= \int_1^2 2\,dt = 2.$$

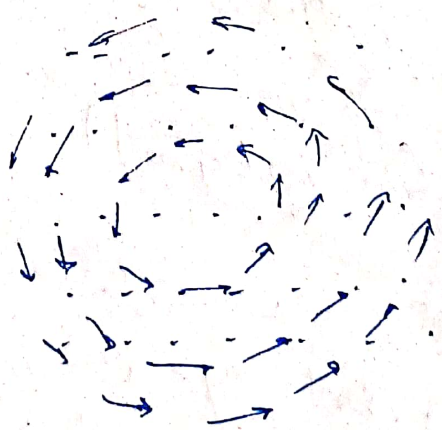
Lec 11 Flow integrals & circulation

$F(x,y) = x\hat{i} + y\hat{j}$



(Dispersing from origin)

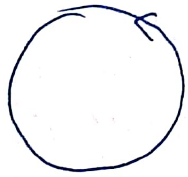
$F(x,y) = -y\hat{i} + x\hat{j}$



(Spin field, everything rotating CCW)

Here we assume vector field as velocity field.

Now imagine you take a path that go through this vector field, Path is same for both



For spin field path aligns with velocity field.

In spin field the path moves along velocity field

In dispersion field the path is orthogonal to velocity field.

* We define Flow : The degree to which vector field is aligned to path.

For a velocity field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ & smooth curve C parameterized by $\vec{r}(t)$

$$\boxed{\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds}$$

$$= \int_a^b F(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_a^b M\,dx + \int_a^b N\,dy.$$

If curve C is closed(ends up exactly where ~~x~~ it started), the flow integral is called as circulation.

$$\text{Circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds$$

Now for our dispersion field with curve

, there is no circulation as $\vec{T}$ is alway $\perp^{er}$, to field vector.

Calculate circulation for both the fields.

Curve C: $x^2+y^2=1$

$\vec{F}(x,y) = x\hat{i} + y\hat{j}$

$\gamma(t) = (\cos t, \sin t)$

$\gamma'(t) = (-\sin t, \cos t)$

$\vec{F}(\gamma(t)) = (\cos t, \sin t)$

$$\text{Circulation} = \oint \vec{F}(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \oint (\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt$$

$$= \oint 0\cdot dt$$

$$= 0$$

Now for

$\vec{F}(x,y) = -y\hat{i} + x\hat{j}$

$\vec{F}(r(t)) = (-\sin t, \cos t)$

Circulation $= \oint_C (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt$

$= \oint (\sin^2 t + \cos^2 t)\,dt$

$= \oint dt = 2\pi$

Also calculate by

$\vec{F}(x,y) = -y\hat{i} + x\hat{j} = \cancel{\cos t} -\sin t\,\hat{i} + \cos t\,\hat{j}$

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

Flow $= \int_a^b M\,dx + \int_a^b N\,dy$

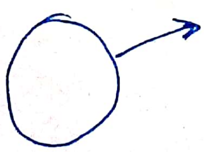
$= \int_a^b (-\sin t)\times(-\sin t)\,dt + \int_a^b \cos t\cdot\cos t\,dt$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi$

Lec-12 Flux Integrals

Till now we saw the field that is tangential to curve

Now, we are interested in normal to curve.



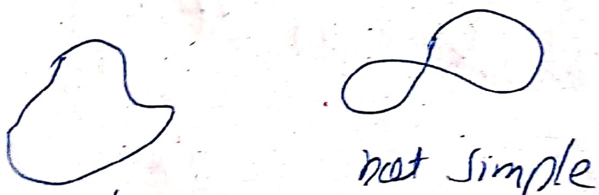
So, the normal flow to curve leads to concept of flux.

For a continuous field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ smooth, simple, closed curve C and $\vec{n}$ the outward normal

$$\boxed{\text{flux} = \oint_C \vec{F}\cdot\vec{n}\,ds}$$

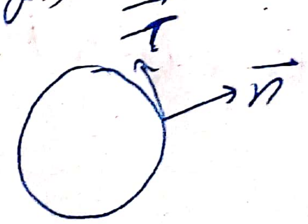
(Here we are talking about 2D vector fields. }, see 3D later)

The field is simple means there are no self intersections.



simple   not simple

(Use simple because the direction definition is necessary.)



We can write $\vec{n}$ as $\vec{T}\times\vec{K}$

($\vec{K}$ is out of plane (paper))

for clo counter clockwise

$$\cancel{\vec{n} = \vec{T} \times \vec{k}}$$

$$\boxed{\vec{n} = \vec{T} \times \hat{k}}$$

$$\vec{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + 0\hat{k}$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} + 0\hat{k}$$

$$\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$$

$$\text{Flux} = \oint_C \vec{F}\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$\boxed{\text{Flux} = \int_a^b M\,dy - \int_a^b N\,dx}$$

Let us calculate flux for dispersion field & spin field.

Dispersion field: $F(x,y) = x\hat{i} + y\hat{j}$

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$ $\quad 0 \le t \le 2\pi$

$$\text{Flux} = \int_0^{2\pi} (\cos t)\times(+\cos t)\,dt - \int_0^{2\pi} \sin t \times (-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi.$$

For spin field

$F(x,y) = y\hat{i} - x\hat{j}$

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$F(r(t)) = \sin t\,\hat{i} - \cos t\,\hat{j}$

$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$$\text{Flux} = \int_0^{2\pi} M\,dy - \int_0^{2\pi} N\,dx$$

$$= \int_0^{2\pi} (\sin t \times \cos t)\,dt - \int_0^{2\pi} ((-\cos t)\times(-\sin t))\,dt$$

$= 0$

For circle $x^2+y^2=1$

$\vec{F} = x\hat{i} + y\hat{j}$	$F = +y\hat{i} - x\hat{j}$
Circulation :	Circulation
$\oint F\cdot T\,ds = 0$	$\oint \vec{F}\cdot\vec{T}\,ds = 2\pi$
Flux	Flux :
$\oint F\cdot\vec{n}\,ds = 2\pi$	$\oint \vec{F}\cdot\vec{n}\,ds = 0$

Lec-13 Conservative Vector field

When A field is conservative on an open domain if $\int_C \vec{F}\cdot d\vec{r}$ is same for all paths C b/w points A & B in the domain.

$\vec{F} = -\hat{j}$ $\quad r(t) = g(t)\,\hat{i} + h(t)\,\hat{j}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(r(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_a^b \langle 0, -1 \rangle \cdot (g'(t), h'(t))\, dt$$

$$= \int_a^b -h'(t)\, dt$$

$$= -(h(b) - h(a))$$

so it depends only on end points, doesn't matter how we get from a to b.

→ When is a field conservative?
→ A continuous field $\vec{F}$ is conservative if & only if

$$\vec{F} = \nabla f$$

for some differentiable 'potential function' f.

Lec 14 Fundamental Theorem of Line Integrals

Fundamental Theorem of calculus (FToC)

If $f'(x)$ is continuous on $[a, b]$

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Fundamental Theorem of line integrals

Let C be smooth curve parameterized by $\vec{r}(t)$ from $\vec{r}(a) = A$ to $\vec{r}(b) = B$

for continuous $\underline{\vec{F} = \nabla f}$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)}$$

Proof:

$$\int_C \vec{F} \cdot dr = \int_a^b F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

By assumption $F = \nabla f$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

By chain rule $= \displaystyle\int_a^b \frac{d}{dt}\{f(\vec{r}(t))\}\, dt$

(By FToC) $= f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(B) - f(A)$.

A continuous field $\vec{F}$ is conservative if & only if

$$\vec{F} = \nabla f$$

For some differentiable 'potential function' f.

Lec15: Testing for conservative vector field

For conservative vector field

1) ~~Pat~~ Path independence
2) Equivalent to $\vec{F} = \nabla f$
3) Fundamental Theorem of line integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

f: potential function.

Two questions:

1) How do we test to see if a field is conservative?
2) How do we find the potential function so $\vec{F} = \nabla f$?

We see each one by one.

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Suppose $\vec{F} = \nabla f$

$$\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow \nabla \times F = 0$$

The field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \; ; \; \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \text{ and } \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}.$$

$(\nabla \times F = 0)$

Lec-16: Finding scalar potential fn. for Conservative vector field

$$\vec{F} = \langle \underbrace{y\cos x + y}_{M}, \underbrace{\sin x + x}_{N}, \underbrace{1}_{P} \rangle$$

$$\frac{\partial M}{\partial y} = \cos x + 1 \qquad \frac{\partial N}{\partial x} = \cos x + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$$

$$\frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}$$

$$\vec{F} = \nabla f$$

$$\frac{\partial f}{\partial x} = y\cos x + y \; ; \; \frac{\partial f}{\partial y} = \sin x + x \; ; \; \frac{\partial f}{\partial z} = 1$$

$$f = y\sin x + yx + C(y,z)$$

$$\frac{\partial f}{\partial y} = \sin x + x + \frac{\partial c}{\partial y} = \sin x + x$$

$$\Rightarrow \frac{\partial c}{\partial y} = 0 \Rightarrow c = c(z)$$

$$\frac{\partial f}{\partial z} = 0 + \frac{dc}{dz} = 1$$

$$\frac{dc}{dz} = 1 \Rightarrow c = z + c'$$

$$f = y\sin x + yx + z + c' \qquad c' = \text{Constant.}$$

Assume $c' = 0$ we get

$$f = y\sin x + yx + z.$$

Now, to compute line integral. we can use $f(b) - f(a)$

Compute $\int_C \vec{F} \, d\vec{r}$ for curve $(0,2,1)$ to $(\pi, 2, 3)$

$$\int_C \vec{F}\cdot d\vec{r} = f(b) - f(a)$$

$$= \cancel{2\sin\theta +}\, f(\pi,1,3) - f(0,2,1)$$

$$= 1\cdot 0 + 1\cdot\pi + 3 - 0 - 0 - 1$$

$$= \pi + 2.$$

(We get answer by easier computation)

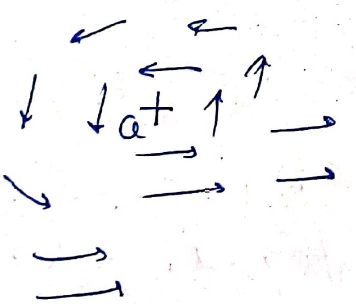
Lec17: Curl or circulation density of a vector field

For the circulation on a curve we had

$$\text{circulation} = \oint_C \vec{F}\cdot d\vec{r}.$$

This was for a curve. This is a global property

Now, there may be ~~a~~ vector fields like



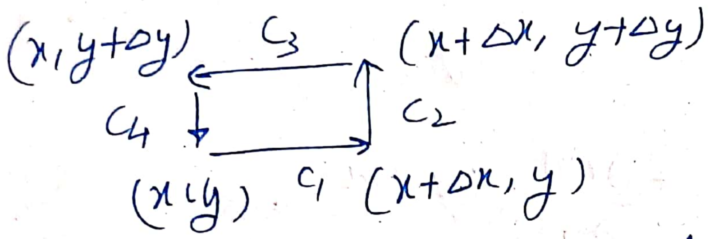
when we look at point a, around it it will be rotating counter-clockwise like →

This is a local property. This is curl at a specific point.

Now, we need to find how much something would spin at a particular point.

Let us ~~zoom~~ in very far.

Let our point be at (x,y), it moves at $(x+\Delta x, y)$, then $(x+\Delta x, y+\Delta y)$, then $(x, y+\Delta y)$ & back to (x,y).



So, we need circulation along the path

$$\oint_C \vec{F}\cdot d\vec{r} = \sum_{i=1}^{4} \int_{C_i} \vec{F}\cdot d\vec{r}$$

$$\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$$

Bottom: $\vec{F}(x,y)\cdot \Delta x\hat{i} = M(x,y)\Delta x$

Right: $\vec{F}(x+\Delta x, y)\cdot \Delta y\hat{j} = N(x+\Delta x, y)\Delta y$

Top: $\vec{F}(x, y+\Delta y)\cdot -\Delta x\hat{i} = -M(x, y+\Delta y)\Delta x$

Left: $\vec{F}(x,y)\cdot -\Delta y\hat{j} = -N(x,y)\Delta y$

(For top & left we go from other dir^n to simplify calculation)

Top + Bottom:

$$M(x,y)\Delta x - M(x, y+\Delta y)\Delta x$$

$$= -\left(M(x, y+\Delta y) - M(x,y)\right)\Delta x$$

We know

$$\frac{\partial M}{\partial y} = \lim_{\Delta y \to 0} \frac{M(x, y+\Delta y) - M(x,y)}{\Delta y}$$

$$\text{Top + Bottom} = -\frac{\partial M}{\partial y}\Delta y \Delta x$$

Right + Left:

$$\cancel{N(x+\Delta x,} \; N(x+\Delta x, y)\Delta y - N(x,y)\Delta y$$

$$= \frac{\partial N}{\partial x}\Delta x \Delta y$$

(only true in limit as Δx & $\Delta y \to 0$)

(It is OK as we are only seeing at a small point)

Adding all

$$\oint_C \vec{F}\cdot d\vec{r} = \frac{\partial N}{\partial x}\Delta y \Delta x - \frac{\partial M}{\partial y}\Delta y \Delta x$$

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$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{\text{Area}}}$$

Circulation around small rectangle

Circulation per unit area (circulation density)

Circulation density is a measure of how much spinning or curling is at a point on your vector field.

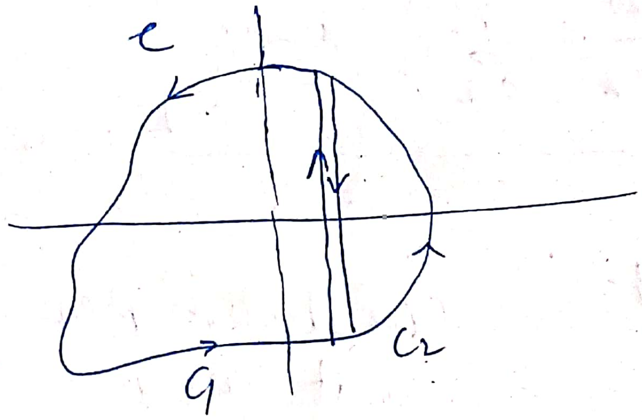
Lec18 Curl, Circulation & Green's Theorem.

At any point we can measure the degree to which the vector field is causing the little propellar to spin at that point, using ~~that~~ circular density.

It is a local property.

Our global property is:

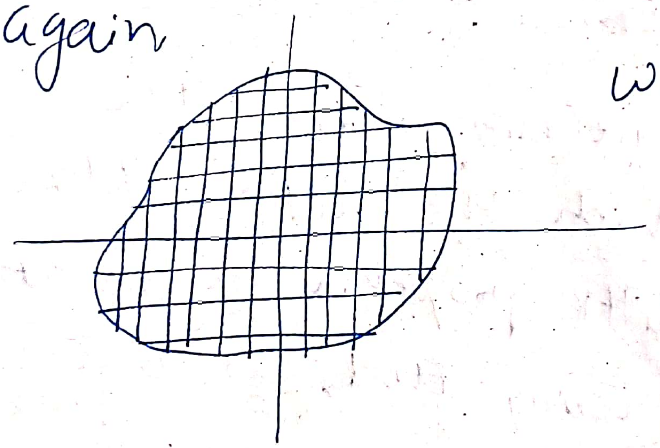
$$\text{Circulation} = \oint_C \vec{F}\cdot d\vec{r}$$



We cut curve into C_1 & C_2.
In C_1 we go up & in C_2 we go down.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$$

In middle portion it cancels.
We can do this over and over again



we have n little regions

So, sum of region outside is equal to sum of all the regions inside as all the internal cut cuts each other.

$$\oint_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n} \oint_{C_i} \vec{F} \cdot d\vec{r}$$

For a small region,

$$\boxed{\not\!\!\approx} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Delta x \Delta y$$

As n goes to infinity

$$\boxed{\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{CCW circulation}} = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} \Delta x \Delta y}$$

Green's Theorem.

We are relating information over entire region and cumalating it all all up and that results in circulation along the boundary.

Lec-19 Divergence, Flux & Green's Thm

We talked about circulation & flow.

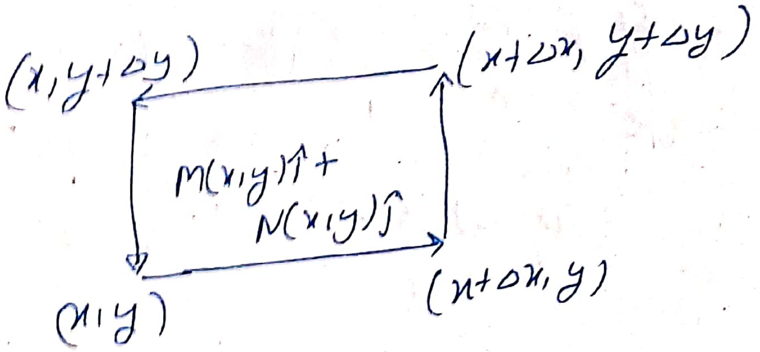
We saw flux $= \oint_C \vec{F}\cdot\vec{n}\, ds.$

we saw in lecture 12:

$$\text{Flux} = \int_a^b M dy - N dx.$$

This was a global property around the curve.

Let us take a very small rectangle,
& in the vector field.
Some vectors are inserting and some
are leaving..



Now, we need what is around the
rectangle.

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds = \sum_{i=1}^{4} \int_C \vec{F}\cdot\vec{n}\, ds$$

$$\int_C \vec{F}\cdot\vec{n}\, ds = \int_a^b M dy - \int_a^b N dx$$

Bottom : $dy = 0 \Rightarrow -N(x,y)\, dx$

Right : $dx = 0 \Rightarrow M(x+\Delta x, y) dy$

Top : $dy = 0 \Rightarrow N(x+\cancel{\Delta x}, y+\Delta y) dx$

Left : $dx = 0 \Rightarrow -M(x, y)\, dy$

$$\text{Total} = M(x+\Delta x, y) dy - M(x,y)\, dy + N(x, y+\Delta y) dx - N(x,y) dx$$

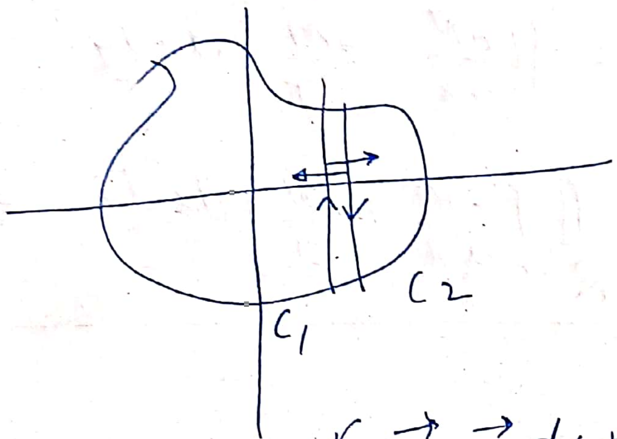
$$= \frac{\partial M}{\partial x} \Delta x \Delta y + \frac{\partial N}{\partial y} \Delta x \Delta y$$

$$\boxed{\oint \vec{F}\cdot\vec{n}\,ds = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y}$$

Flux around small rectangle — Flux density — Area

Flux density is also called divergence. Notion of how much vector field is spreading from the point..

Now, in the same way, as we did for circulation



$$\oint \vec{F}\cdot\vec{n}\,ds = \oint_{C_1} \vec{F}\cdot\vec{n}\,ds + \oint_{C_2} \vec{F}\cdot\vec{n}\,ds.$$

Everything cancels at the cut. We can cut more and more

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \sum_{i=1}^{n} \oint_{C_i} \vec{F}\cdot\vec{n}\,ds$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

We get Green's Theorem in divergence - flux form

$$\boxed{\underbrace{\oint_C \vec{F}\cdot\vec{n}\,ds}_{\text{outward flux}} = \iint_R \underbrace{\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)}_{\text{Divergence}} dx\,dy}$$

Lec-20

Green's Theorem

$$\boxed{\begin{aligned} &1)\quad \oint_C \vec{F}\cdot\vec{T}\,ds = \iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \\ &2)\quad \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R\left[\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right]dx\,dy \end{aligned}}$$

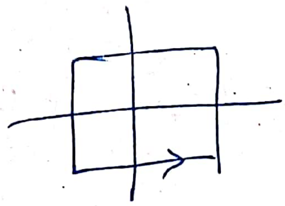
$\oint_C \vec{F}\cdot\vec{T}\,ds =$ CCW circulation

$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} =$ curl / circulation density

$\oint_C \vec{F}\cdot\vec{n}\,ds =$ Flux

$\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y} =$ Divergence

Ex: Let $\vec{F} = x^2\hat{i} + xy\hat{j}$ on square $x = \pm 1$ and $y = \pm 1$. Compute circulation & flux.



$\Rightarrow$ To calculate circulation we use the curl approach as using line integral will involve, breaking into 4 parts.

$$f = (\underset{M}{x^2}, \underset{N}{xy})$$

$$\oint \vec{F}\cdot\vec{T}\,ds = \iint (y - 0)\,dx\,dy \qquad \begin{array}{l} x \in [-1, 1) \\ y \in [-1, 1] \end{array}$$

$$= \int_{y=-1}^{1}\int_{x=-1}^{1} y\,dx\,dy$$

$$= 2\int_{-1}^{1} y\,dy = 0.$$

Let us do line integrals

Bottom: $\Delta y = 0 \quad \oint \vec{F}\cdot \cancel{dr}\, r'(t)\,dt$

$$r(t) = (t, -1, 0) \qquad t \in [-1, 1]$$

$$= \cancel{t^2\hat{i}} \quad r'(t) = (1, 0, 0)$$

$$\cancel{F(t)} = F(r(t)) = (t^2, -t)$$

$$C_1: \int_{C_1} \vec{F}\cdot r'(t)\,dt = \int_{-1}^{1} t^2\,dt = \left|\frac{t^3}{3}\right|_{-1}^{1} = 2/3.$$

C_2 : $\gamma(t) = (1, t, 0)$ $t \in [-1, 1]$

$\gamma'(t) = (0, 1, 0)$

$F(x,y) = (x^2, xy)$ $t \in [-1, 1)$

~~$F(t) =$~~

$F(\gamma(t)) = (1, t, 0)$

$\int_{C_2} F \cdot \gamma'(t)\, dt = \int_{-1}^{1} t \, dt = \left| \frac{t^2}{2} \right|_{-1}^{1} = 0$

C_3 : $\gamma(t) = (t, 1, 0)$ $t \in [1, -1]$

$\gamma'(t) = (1, 0, 0)$

$F(\gamma(t)) = (t^2, t, 0)$

$\int_{C_3} F \cdot \gamma'(t)\, dt = \int_{+1}^{-1} t^2 \, dt = \left| \frac{t^3}{3} \right|_{+1}^{-1}$

$= -2/3.$

C_4 : $\gamma(t) = (-1, t, 0).$ $t \in [1, -1)$

$\gamma'(t) = (0, 1, 0)$

$F(\gamma(t)) = (1, -t, 0)$

$\int_{C_4} F \cdot (\gamma'(t))\, dt = \int_{1}^{-1} -t \, dt = 0.$

Total $= \frac{2}{3} + 0 - \frac{2}{3} + 0 = 0.$

And from curl approach also we get 0.

Now, let us see for flux.

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

$$F = (x^2, xy)$$

$$M = x^2, \quad N = xy$$

$$\iint (2x + x)\, dx\, dy \qquad \begin{array}{l} x \in [-1, 1] \\ y \in [-1, 1] \end{array}$$

$$\iint 3x\, dx\, dy = \cancel{\frac{3x^2}{2}\Big|_{-1}^{1}}$$

$$\int 3x\, |y|_{-1}^{1}\, dx$$

$$= \int_{-1}^{1} 6x\, dx = 0.$$

We get flux also equal 0.

Let us calculate using line integral.

$$\oint \vec{F}\cdot\vec{n}\, ds$$

$$\vec{n} = \vec{T} \times \hat{k} = \cancel{r'(t) \times \hat{k}}$$

$$\vec{T} = \frac{d\vec{r}}{ds}$$

$$\oint \vec{F}\cdot\vec{n}\, ds = \oint \vec{F}\cdot\left(\frac{d\vec{r}}{ds} \times \hat{k}\right) ds$$

$$= \oint \vec{F}\cdot\left(\frac{d\vec{r}}{ds} \times \hat{k}\right) \frac{ds}{dt} \times dt$$

$$\boxed{\oint \vec{F}\cdot\vec{n}\, ds = \oint \vec{F}\cdot\left(r'(t) \times \hat{k}\right) dt}$$

Now we have in one parameter.

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$$\oint \vec{F}\cdot\vec{n}\,ds = \oint \vec{F}\cdot(r'(t)\times\hat{k})\,dt$$

$F = (x^2, xy)$

For C_1 : $r(t) = (t, -1, 0)$ $t \in [-1, 1]$

$r'(t) = (1, 0, 0)$

$$r'(t)\times\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \cancel{0}-\hat{j} = -\hat{j}$

$F(r(t)) = (t^2, -t)$

$\oint F(r(t))\cdot(r'(t)\times\hat{k})\,dt =$

$\int_{t=-1}^{1} (t^2, -t)\cdot(0, -1)\,dt$

$= \int_{-1}^{1} t\,dt = 0.$

For C_2 : $r(t) = (1, t, 0)$ $t \in [-1, 1]$

$r'(t) = (0, 1, 0)$

$r'(t)\times\hat{k} = (1, 0, 0)$

$F(r(t)) = (1, t)$

$\oint_{C_2} F(r(t))\cdot(r'(t)\times\hat{k})\,dt =$

$\int_{t=-1}^{1} (1, t)\cdot(1, 0)\,dt$

$= \cancel{(t)\Big|_{-1}^{1}} = 2.$ $\int_{-1}^{1} 1\,dt = (t)\Big|_{-1}^{1}$

$= 2$

For C_3 : $\gamma(t) = (t, 1, 0)$ $\quad t \in [1, -1]$

$\gamma'(t) = (1, 0, 0)$

$\gamma'(t) \times \hat{k} = -\hat{j}$ $\quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$F(\gamma(t)) = (t^2, t)$

$$\int_{C_3} F(\gamma(t)) \cdot (\gamma'(t) \times \hat{k})\, dt = \int_1^{-1} -t\, dt = 0$$

For C_4 : $\gamma(t) = (-1, t, 0)$ $\quad t \in [1, -1]$

$\gamma'(t) = (0, 1, 0)$

$\gamma'(t) \times \hat{k} = (1, 0, 0)$

$F(\gamma(t)) = (1, -t)$

$$\int_{C_4} F(\gamma(t)) \cdot (\gamma'(t) \times \hat{k})\, dt = \int_{+1}^{-1} (1, -t) \cdot (1, 0)\, dt$$

$$= \int_1^{-1} 1\, dt$$

$$= -2$$

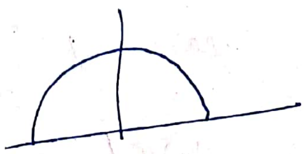
Summing all $= 0 + 2 + 0 - 2$

$= 0.$

So, we get 0 from both the methods.

Lec-21 Describing surfaces explicitly, implicitly and parametrically

We can describe our curve



Implicitly: $x^2 + y^2 = 1,\ y > 0$

Explicitly : $y = \sqrt{1 - x^2}$

Parametric: $\bar{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j}$

$0 \le t \le \pi$

For surfaces in 3D

Implicit: $F(x,y,z)=c$

Explicit: $z=f(x,y)$

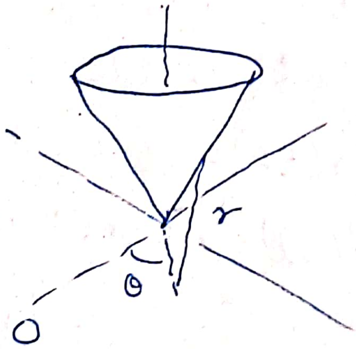
Parametric:

$$\vec{r}(u,v)=f(u,v)\hat{i}+g(u,v)\hat{j}+h(u,v)\hat{k}$$

$$a\le u\le b;\quad c\le v\le d$$

(We have 2 parameters u,v)

For cone



Explicit: $z=\sqrt{x^2+y^2}$

Implicit: $z-\sqrt{x^2+y^2}=0$

Parametric: $\vec{r}(x,y)=x\hat{i}+y\hat{j}+\sqrt{x^2+y^2}\,\hat{k}$.

Parameter: (x,y)

We can also use r,θ.

Explicit $z=\sqrt{x^2+y^2}=r$ $\qquad z\le 3$ (restriction)

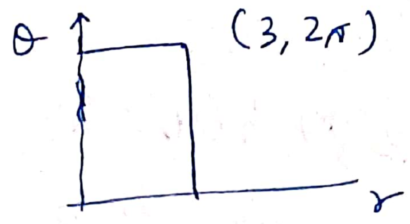
Parametric:

$$\vec{r}(r,\theta)=r\cos\theta\,\hat{i}+r\sin\theta\,\hat{j}+r\hat{k}$$

We are ~~try~~ trying to describe a 2D surface.

For restriction, $0\le r\le 3$. & $\theta\in[0,2\pi]$.

So, in the r-θ coordinate system the cone is just a rectangle



o

So, we can transform a simple region in r-θ to a more 2D complicated 2D region that is the cone embedded in 3D.

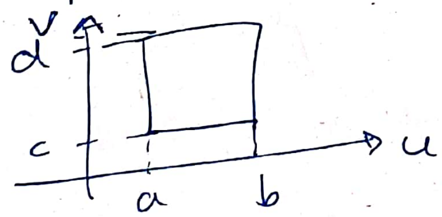
So, r-θ we are able to simplify the process..

———x ———x ——x —— x—— x——

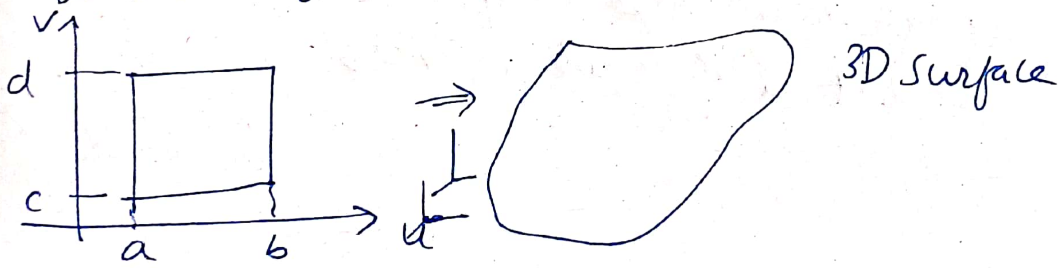
Lec-22 Surface area for parametric surfaces

We will see surfaces that are described parametrically.

We have a surface lik- 

We write it parametrically means.. We have a region 

So, we have a space in u-v. Such that there is some map in 2D region (UV plane) to the larger area described above

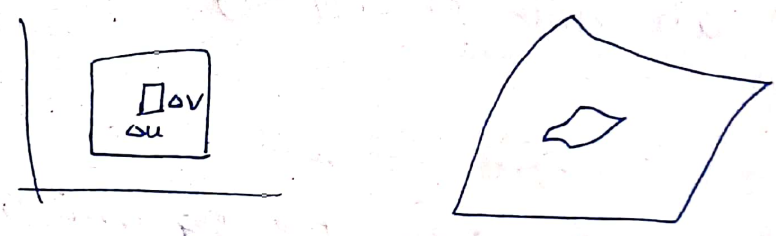


So, we write

$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$.

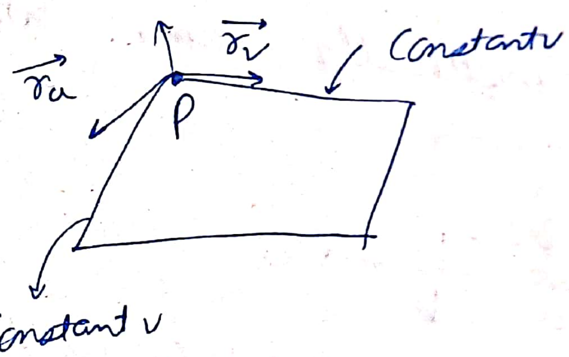
So, we transform from u-v region to a larger surface that lives in 3D despite itself being 2D.

We solve for a small region.



The region on 3D surface ~~is~~ represent boundary where one side is something with a constant value of Δu & other side is something with a constant value of Δv.

We take this small surface

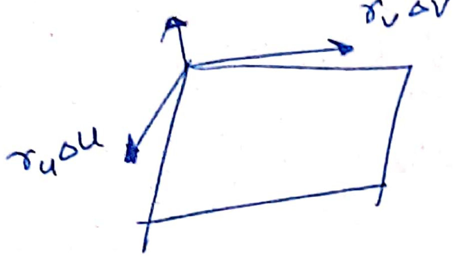


$\vec{r_u}$ points in dir^n of constant v.

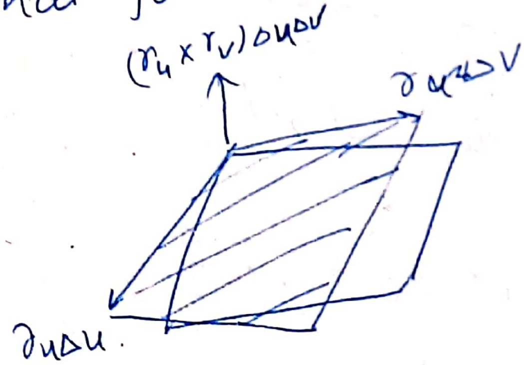
$\vec{r_v}$ points in dir^n of constant u.

We zoom in large enough to consider constant u & v are st. lines.

We have a third vector
$\vec{r_u} \times \vec{r_v}$
We ~~k~~ also need a scaling factor.
In dir^n of $\vec{r_u}$, scaling is Δu. & in dir^n of $\vec{r_v}$, scaling is Δv.
And our 1st vector will be ~~$(\vec{r_u} \times \vec{r_v}) \Delta u \Delta y$~~
$(r_u \times r_v) \Delta u \Delta v$



The length of cross product is the area of the parallelogram of the two vectors that form the cross product.



Area of parallelogram
$= |\vec{r_u} \times \vec{r_v} \Delta u \Delta v|$

We still don't know exact surface area of the original little portion of the surface. We approximate it with area of parallelogram.
Area $= |r_u \times r_v| \Delta u \Delta v$.
As we make it small & small, it transforms to definition of integral.

$$\boxed{\text{Surface Area} = \iint_{v\,u} |r_u \times r_v|\, du\, dv}$$

We can compute surface area by taking cross product and multiplying by the small scaling factor Δu Δv & integrating over the whole surface.

Lec-23 Surface area of a sphere

Implicit : $x^2+y^2+z^2=a^2$

Parameterization:

$r(\phi, \theta) = (a \sin\phi \cos\theta, a\sin\phi \sin\theta, a\cos\phi)$

$a =$ fixed

$0 \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$

Surface area $= \int_a^b \int_c^d |r_u \times r_v|\, du\, dv$

$r_\phi = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi)$

$r_\theta = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0)$

$r_\phi \times r_\theta = a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi$

$|r_\phi \times r_\theta| = a^2\sin\phi.$

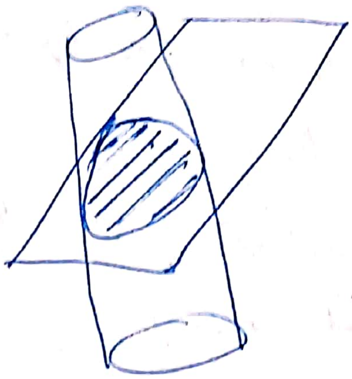
$$A = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi \, d\phi\, d\theta = a^2\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta$$

$$= 4\pi a^2$$

Lec-24

Find surface area of plane $z = -x$ inside $x^2 + y^2 = 4$

$\Rightarrow$



we need area of shaded region.

$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ -r\cos\theta)$

$(z = -x)$

$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ -r\cos\theta)$

$\Downarrow$

$z = -x$ (given)

$0 \le r \le 2,\quad 0 \le \theta \le 2\pi \quad (x^2 + y^2 = 4)$

$$SA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} |\vec{r}_r \times \vec{r}_\theta|\, dr\, d\theta$$

$\vec{r}_r = (\cos\theta,\ \sin\theta,\ -\cos\theta)$

$\vec{r}_\theta = (-r\sin\theta,\ r\cos\theta,\ r\sin\theta)$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$\hat{i}(r) - \hat{j}(0) + \hat{k}(r)$

$= r(\hat{i} + \hat{k})$

$|\vec{r}_r \times \vec{r}_\theta| = r\sqrt{2}$

$$SA = \int_0^{2\pi}\int_0^{2} \sqrt{2r^2}\, dr\, d\theta$$

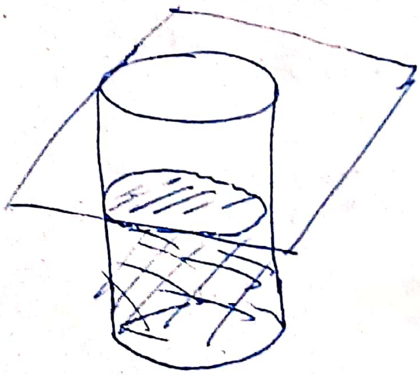
(Here we can't put $r\,dr\,d\theta$, as this is not conversion from $dx\,dy$ to $r\,dr\,d\theta$. Our formula is for parameterized surface, so we just integrate over parameters. Whole integration here is in $r\theta$ space)

$$SA = \int_0^{2\pi}\int_0^{2} \sqrt{2}\, r\, dr\, d\theta$$

$$= 4\sqrt{2}\,\pi.$$

Lec-25

Find surface area of $x^2+y^2=4$ b/w $z=0$ & $z=16-2x$.



We ~~no~~ need the area ##

$$SA = \iint_R |r_u \times r_v|\, du\, dv$$

$r(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$\theta \in [0, 2\pi]$

$r_\theta = (-2\sin\theta, 2\cos\theta, 0)$

$r_z = (0, 0, 1)$

$r_\theta \times r_z = -\cancel{2\sin\theta}\,(2\cos\theta, 2\sin\theta, 0)$

$|r_\theta \times r_z| = 2.$

$SA = \iint 2\, d\theta\, dz$

$z = 0$ to $16 - 2x$
$\Rightarrow \cancel{16 - 2\cos\theta}$
$= 16 - 2\times 2\cos\theta$
$= 16 - 4\cos\theta$

$$= \int_0^{2\pi} \int_{z=0}^{16-4\cos\theta} 2\, d\theta\, dz$$

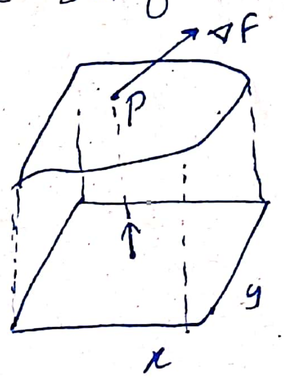
$$= 2\int_0^{2\pi} (16 - 4\cos\theta)\, d\theta$$

$$= 2\,(16\theta - 4\sin\theta)\Big|_0^{2\pi}$$

$$= 2\times 16\times 2\pi = 64\pi.$$

Lec-26: Surface area for implicit & explicit surfaces

Implicit surface: $F(x,y,z) = c$



Consider a plane in x-y
Now there is a surface above the xy plane.
~~Cons~~ Consider a point P in the above plane.
We have ∇F at that point, it is normal to surface at that point.

Corresponding to the point P, we have a point in x-y plane, It will point in dir^n of $\hat{k}$.

We are talking about surfaces such that

$\nabla f \cdot \hat{k} \neq 0$

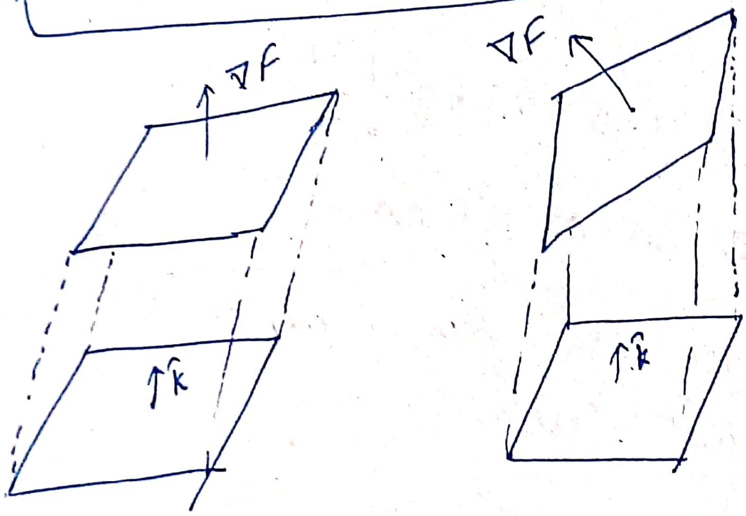
Implicit surface $F(x,y,z)=c$

Assume:

- Smooth (i.e. F differentiable, $\nabla F \neq 0$ & continuous)
- The surface is above a region in the x-y plane with $\nabla F \cdot \hat{k} \neq 0$

$$\boxed{\text{Surface Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{k}|}\, dA}$$

dA = area of the x-y plane. (that is projected)



$\dfrac{\nabla F}{|\nabla F \cdot \hat{k}|}$ is the stretching factor

The more plane is twisted, $|\nabla F \cdot \hat{k}|$ decreases so it is bigger overall.

Now instead of assuming plane above x-y plane, we can assume the surface above a plane region with $\nabla f \cdot \hat{p} \neq 0$

$$\boxed{\text{Surface Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} \, dA}$$

$\vec{p}$ is $\hat{i}$ or $\hat{j}$ or $\hat{k}$.

For explicit surface $z = f(x,y)$

set $F(x,y,z) = f(x,y) - z = 0$

$\nabla F = f_x \hat{i} + f_y \hat{j} - \hat{k}$

$|\nabla F \cdot \hat{k}| = |-1| = 1$

$|\nabla F| = \sqrt{1 + f_x^2 + f_y^2}$

$$\text{Surface Area} = \iint \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} \, dA$$

$$= \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA$$

Lec-27

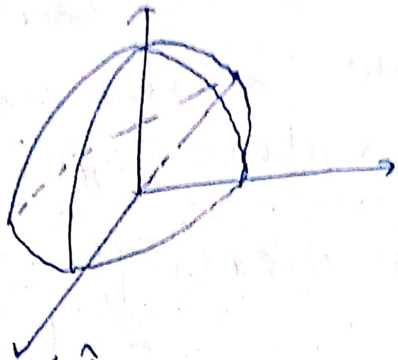
Surface Area of $z + x^2 + y^2 = 1$ above x-y plane

$F(x,y,z) : z + x^2 + y^2 = c = 1$

$$SA = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{k}|} \, dA$$

$z + x^2 + y^2 = 1$

$z = 1 - x^2 - y^2$



$F: z + x^2 + y^2 - 1 = c$

$\nabla F = 2x\hat{i} + 2y\hat{j} + 1\hat{k}$.

$|\nabla F| = \sqrt{1 + 4x^2 + 4y^2}$

$\nabla F \cdot \hat{k} = 1 \qquad |\nabla F \cdot \hat{k}| = 1$

$$SA = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA$$

For dA, we need region. The region in x-y plane is a circle $x^2 + y^2 = 1$. $(z = 0)$

So, we integrate in polar coordinates

$$SA = \iint_R \sqrt{1 + 4r^2}\; r\,dr\,d\theta \qquad r \in [0, 1], \quad \theta \in [0, 2\pi]$$

$u = 1 + 4r^2$

$du = 8r\,dr$

$$SA = \frac{1}{8} \iint u^{1/2}\, du\, d\theta$$

$$= \frac{1}{8} \times \int_0^{2\pi} \frac{2}{3}(4r^2 + 1)^{3/2} \Big|_{r=0}^{1} d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3}\left((5)^{3/2} - 1\right) d\theta$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

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Lec 28 Surface Integrals

Previously we talked about surface area now we will talk about surface integrals

A smooth surface S has surface area

$$\iint_S d\sigma$$

Parametric:

$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

$$SA = \iint_R |\vec{r_u} \times \vec{r_v}| \, du \, dv$$

Implicit : $F(x,y,z) = c$

$$SA = \iint_R \frac{\nabla F}{|\nabla F \cdot \vec{p}|} dA$$

Explicit : $z = f(x,y)$

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA$$

Surface Integral $= \iint_S G(x,y,z) \, d\sigma$

Now $G(x,y,z)$ defined on a smooth surface S.

Parametric

$$SI = \iint_R G(f(u,v), g(u,v), h(u,v)) \, |\vec{r_u} \times \vec{r_v}| \, du \, dv$$

Implicit :

$$SI : \iint G(x,y,z) \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dA$$

Explicit

$$SI : \iint_R G(x,y,f(x,y)) \sqrt{1 + f_x^2 + f_y^2} \, dA$$

Application #1: Mass of shells

Let $\delta(x,y,z)$ be the mass density of a thin shell S

$$\text{Total mass} = \iint_S \delta(x,y,z)\, d\sigma$$

Application #2: Averages

(made up fn)

Let $T(\phi,\theta) = -20 + 50\sin\phi$ be the temperature on a sphere of radius a.

$\Rightarrow$ Temp at poles is -20, and temp at equator is $-20+50 = 30$.

$$\text{Average temp} = \frac{1}{4\pi a^2} \iint_S T(\phi,\theta)\, d\sigma$$

$$= \frac{1}{4\pi a^2} \int_0^{2\pi}\int_0^{\pi} (-20 + 50\sin\phi)\underbrace{a^2 \sin\phi}_{|\vec{r}_\phi \times \vec{r}_\theta|}\, d\phi\, d\theta$$

$$= \frac{a^2}{4\pi a^2} \int_0^{2\pi}\int_0^{\pi} \left(-20\sin\phi + \frac{50}{2}(1-\cos(2\phi))\right) d\phi\, d\theta$$

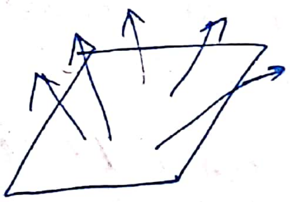
$$= \frac{2\pi a^2 (25\pi - 40)}{4\pi a^2} = \frac{25\pi - 40}{2} \approx 18°C$$

The average temp is 18°C (made up function).
The minimum was $-20°$, & max was 30°C.
18°C is much closer to 30°C, this is because surface area ~~across~~ around the pole is

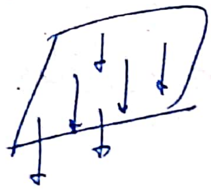
quite less, while surface area around the equator is quite large. So, the average gets weighted by the area.

Lec 29 Orientable & non-orientable surfaces

Def^n^: A smooth surface is orientable if there is a continuous field of unit normal vectors.

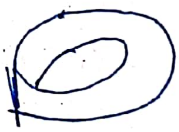


all outward



all inward.

This is not always possible ex in mobius strip. It only has one side.



Normal paper has two side but mobius strip has only 1 side.

~~When~~ The normal vector change the orientation after 360° at the ~~p~~ same point. So, it is non-orientable.

We ~~st~~ restrict ourself to orientable surfaces.

$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} \qquad t \in [a,b]$

A

B

The arrow shows orientation of the curve.

If we have an orientable surface, then we need to know which way it is oriented, so that we can use the normal vector.

Lec-~~29~~ 30 Flux across a ~~stu~~ surface

We have a surface and a vector field that swirling around or across that surface according to arrows in vector field.

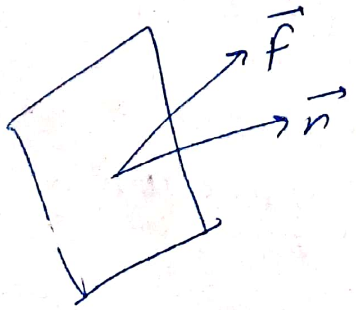
Flux: Degree to which vector field crosses the surface

We saw

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n} \, ds \quad (\vec{n} \text{ is outward normal})$$

We assume that surface is orientable, so we have two choice for normal vector. (inward & outward)

Now, we need to find how much vector field is crossing the boundary in outward dirn. For a small surface



$\vec{F} \cdot \vec{n}$ is proportion of field in the normal direction

For a continuous field $\vec{F}$ on a smooth surface S oriented with normal unit vectors $\vec{n}$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

Parametric:

$$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$$

$$\text{Unit normal} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$\Rightarrow \text{Flux} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$\text{Flux} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

Implicit: $g(x,y,z) = c$

(Taking $\vec{F}$ as vector field)

$$\text{Unit normal} = \frac{\nabla g}{|\nabla g|}$$

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} \, du \, dv$$

$$\text{Flux} = \iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} \, du \, dv$$

$$= \iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{p}|} \, du \, dv$$

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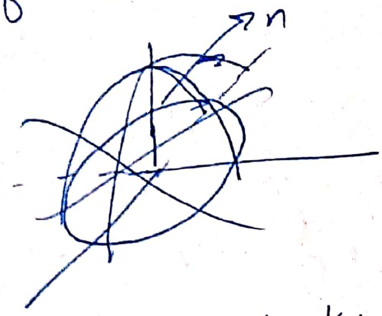
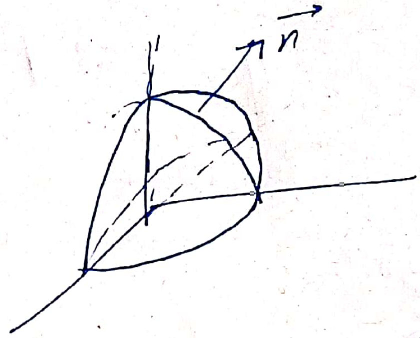
Lec-~~30~~31

Compute flux across $z = 1 - x^2 - y^2$, $z \geq 0$ of $\vec{F} = \langle x, y, z \rangle$





We are taking outward normal.

$\vec{F} = \langle x, y, z \rangle$

Flux $= \iint \vec{F} \cdot \vec{n} \cdot d\sigma$

We can use parametric curve

$= \iint \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$

$\vec{r} = \langle r\cos\theta, r\sin\theta, 1 - r^2 \rangle$

$\vec{F} = \langle r\cos\theta, r\sin\theta, 1 - r^2 \rangle$

(r can't be 1, as we need flux around the paraboloid surface r varies from 0 to 1 & θ from 0 to 2π)

$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= 2r^2\cos\theta\, \hat{i} + \hat{j}\, 2r^2\sin\theta + r\,\hat{k}$

$$\text{Flux} = \iint \langle r\cos\theta,\ r\sin\theta,\ 1-r^2\rangle \cdot \langle 2r^2\cos\theta,\ 2r^2\sin\theta,\ r\rangle\, dr\,d\theta$$

$$= \iint \left(2r^3\cos^2\theta + 2r^3\sin^2\theta + r - r^3\right) dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r + r^3)\, dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^1 d\theta = 2\pi \times (3/4)$$

$$= 3\pi/2.$$

Lec-~~31~~32 Curl of a vector field

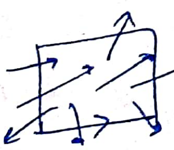
Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be vector field.

$$\text{curl } \vec{F} = \left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right]\hat{i} + \left[\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right]\hat{j} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\hat{k}.$$

circulation density.

We saw circulation density in 2D in lec-18, 17.

For a small rectangle.

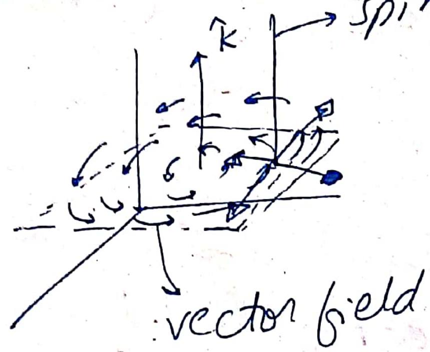


vector fields

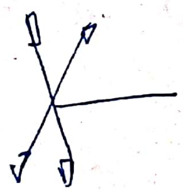
The flow around this ~~area~~ curve per unit area was expressed by $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\Delta y$.

We are trying the measure the degree to which the vector field is curling at that particular point in the limit as the loop gets smaller and smaller.

Curl is the 3D analog of this 2D circulation density.



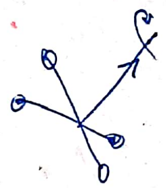
In the plane $\parallel^{el}$ to x-y plane, we have a vector field. We put a spinner in that vector field. Now how much would that spin is given by our third component $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ (as it is $\parallel^{el}$ to x-y plane)



Now, spinner y-z plane,
So spin is given by
$\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}$ ie. 1^{st} component.

Now if we put spinner in the vector field, and fix it to a point. It would orient itself and point towards a

certain direction & then spins a m.
So, the dir^n towards it points is called as curl.



$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

~~Curl~~ $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ (gradient)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = \text{curl } \vec{F}.$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= [f_{zy} - f_{yz}]\hat{i} + [f_{zx} - f_{xz}]\hat{j} + [f_{yz} - f_{zy}]\hat{k}$$

$= \vec{0}$ if f has continuous mixed 2nd derivatives.

Recall:

$\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F} = \nabla f$

$\Rightarrow \nabla \times \vec{F} = 0$

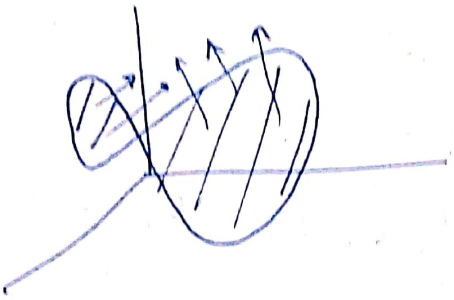
For conservative field $\boxed{\nabla \times \vec{F} = \vec{0}.}$

$\Updownarrow$

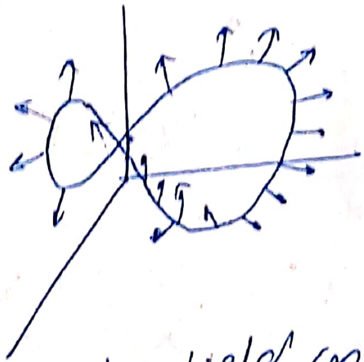
Conservative

Lec-33 Stoke's Theorem

We have a surface in a vector field.
Now we imagine boundary curve of the surface & vector fields around those boundary curve

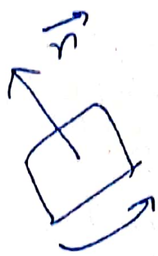


vector field on surface



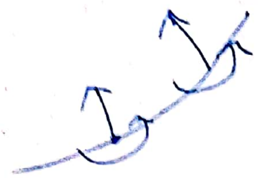
vector field on the curve

We are focused on boundary because we studied circulation along the boundary by this vector field $= \oint_C \vec{F} \cdot d\vec{r}$



Focus on small ~~cu~~ surface. There will be normal vector and curl that will rotate it ~~a~~ in the region.
This is true for any point on the surface.

Now, if we go to any point on the boundary. There will be normal & circulation around that boundary.



At any point : Curl $= \nabla \times \vec{F}$

$(\nabla \times \vec{F}).\vec{n} \Rightarrow$ curl within the surface.

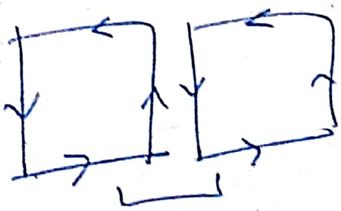
i.e. curl subject to the constraint that we must remain on the surface.

We know

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{CCW circulation}} = \iint \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} dx\,dy$$

$$= \iint \underbrace{(\nabla \times \vec{F})\cdot\hat{k}}_{k^{th}\text{ component of curl.}}\, dx\,dy$$

We know that in interior all cancels out and we are left with all around the boundary.



cancels.

Now for a 3D surface

$$\boxed{\oint \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot \vec{n} \, d\sigma}$$

Stokes Theorem

CCW circulation   curl.

Now our vector field is 3D not 2D, and surface is 3D, not 2D.

* Instead of $dx\,dy$ we have $d\sigma$ that is the surface integral.

(Curl lives inside the surface, we add all the curls, it inside one cancels and we are left with curling around the boundary)

Conditions:- For S a piecewise smooth oriented ~~smooth~~ surface with ~~continuous~~ piecewise smooth boundary. C and $\vec{F}$ a field with continuous first partials for each component on an open region containing S.

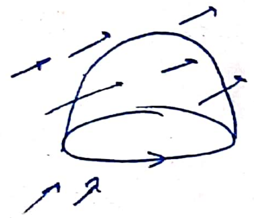
Lec-34 Example

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0$$

$$\vec{F} = y\hat{i} - x\hat{j} + z\hat{k}$$

We have a hemisphere $x^2+y^2+z^2=1$ $z \geq 0$
And we have a vector field

$$\vec{F} = y\hat{i} - x\hat{j} + z\hat{k}$$



we need orientation for surface and the boundary curve. that should be compatible with each other.
We imagin the boundary curve is roted CCW, ~~and~~ when viewed from above. This is compatible with having normal vectors to the surface be pointing outwards ~~in~~ at any particular point.

$\oint \vec{F} \cdot d\vec{r}$ = 

$\iint (\nabla \times F) \cdot \vec{n}\, d\sigma$ = 

One may be easy, other may be not. We see by both formula.

① $\oint_C \vec{F} \cdot d\vec{r}$  xy plane.

$\vec{r}(\theta) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k}$

$r'(\theta) = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\vec{F}(r(\theta)) : \sin\theta\,\hat{i} - \cos\theta\,\hat{j} + 0\,\hat{k}$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} \langle \sin\theta, -\cos\theta\rangle \cdot \langle -\sin\theta, \cos\theta\rangle \, d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta - \cos^2\theta)\, d\theta$$

$$= -2\pi.$$

Now, we see right side

$$\iint_S (\nabla\times F)\cdot \vec{n}\, d\sigma$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & z \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} - 2\hat{k}.$$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} \qquad f: x^2+y^2+z^2=1$$

$$\vec{n}\, d\sigma = \underbrace{\frac{\nabla f}{|\nabla f\cdot\hat{k}|}}_{\text{Stretching factor}} dA$$

for plane area (projected)

(Lec-26)

$$\iint_R (\nabla\times F)\cdot\left(\frac{\nabla f}{|\nabla f\cdot\hat{k}|}\right) dA$$

$$\nabla f = (2x\hat{i}, 2y\hat{j}, 2z\hat{k})$$

$$\nabla f\cdot\hat{k} = 2z$$

$$\frac{\nabla f}{|\nabla f\cdot\hat{k}|} = \frac{\langle 2x, 2y, 2z\rangle}{2z}$$

$$\iint_R (\nabla \times F) \cdot \frac{\nabla f}{|\nabla f \cdot \hat{k}|} \, dA$$

(we shift from $\underline{S}$ in $\iint_{\underline{S}} \nabla \times F \cdot \vec{n} \, dA$ to $\underline{R}$ in

$$\iint_{\underline{R}} (\nabla \times F) \cdot \frac{\nabla f}{|\nabla f \cdot \hat{k}|} \, dA \;)$$

$\nabla \times \vec{F} = -2\hat{k}$

$$\iint (-2\hat{k}) \cdot \left(\frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2z} \right) dA$$

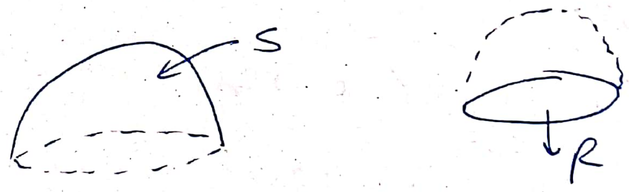
$$\iint -\frac{4z}{2z} \, dA$$

$$= \iint -2 \, dA$$

$$= -2 \iint dA$$

area of circle of radius $1 = \pi (1)^2 = \pi$

$$= -2\pi.$$



We find that surface integral of a surface ~~λ~~ is enclosed by a boundary of region R is same as line integral. So, now if we have same boundary of any surface ~~area~~ area ~~areas~~ in that boundary, will be equal to any other surface area enclosed by that boundary.

Like the surface enclosed in the region R can be a disc.

So, for a disc

$$\iint (\nabla \times F) \cdot \vec{n} \, d\sigma \qquad d\sigma = dA$$

$$= \iint (-2\hat{k}) \cdot \hat{k} \, \cancel{dA} \, dA$$

$$= \iint -2 \, dA$$

$$= -2\pi .$$

So, now we don't have to use the surface that is given we can use any other surface as long as it has same boundary. So, we can make the easiest surface that we can imagine. So, we can have the flat region inside the curve.

Lec 35 Divergence Theorem

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$.

$$\boxed{\text{Div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}} = \nabla \cdot \vec{F}$$

We saw divergence during flux. If we zoomed very far in, then for a small rectangle, what was the degree to which the field was crossing across the boundary is called divergence of field / flux density of field.

Divergence / flux Density: $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

We saw Greens theorem in divergence-flux form

$$\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

(Stokes theorem is generalization of one half of greens theorem)

(Divergence theorem is the generalization of other half of green's theorem)

$$\underbrace{\oint_C \vec{F} \cdot \vec{n} \, ds}_{\text{outward flux}} = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)}_{\text{divergence}} dA$$

(add local & it cancels out to get value at boundary)

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \vec{F}$$

Now, we apply it to 3D

$$\boxed{\underbrace{\iint_S \vec{F} \cdot \vec{n} \, ds}_{\text{outward flux}} = \iiint_D \underbrace{\nabla \cdot \vec{F}}_{\text{Divergence}} \, dV}$$

Now, we are seeing outward flux across a surface

$$\iint_S \vec{F} \cdot \vec{n}\, ds = \iiint_D \nabla \cdot \vec{F}\, dV$$

Condition: For S a piecewise smooth oriented closed surface with piecewise smooth boundary C and, $\vec{F}$ a field with continuous first partials for each component on an open region containing S.

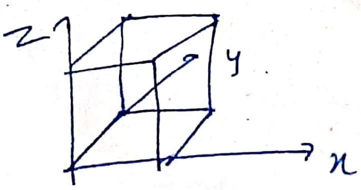
In greens theorem we demanded that curves were closed and that contained an area. Now we imagine that our surface is closed and thus containing a volume.

Lec-36

Example

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$$

Flux of $\vec{F} = \langle xyz, xyz, xyz \rangle$ out of unit ~~cube~~ cube.



unit cube

How instead of using 6 different integrals for each of 6 different surface. We will use one 1 integral.

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot \vec{F} = yz + xz + xy$$

$$\iiint (yz + xz + xy)\, dx\, dy\, dz$$

$x \in [0,1]$, $y \in [0,1]$, $z \in [0,1]$

$$\iint \left(xyz + \frac{x^2}{2}z + \frac{x^2}{2}y\right)\Big|_{x=0}^{1} dy\, dz$$

$$= \iint \left(yz + \frac{z}{2} + \frac{y}{2}\right) dy\, dz$$

$$= \int \left(\frac{y^2}{2}z + \frac{zy}{2} + \frac{y^2}{4}\right)\Big|_0^1 dz$$

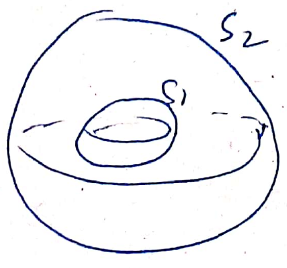
$$= \int \left(\frac{z}{2} + \frac{z}{2} + \frac{1}{4}\right) dz$$

$$= \left. \frac{3}{4}\frac{z^2}{2} + \frac{1}{4}z \right|_0^1 = \frac{3}{4}$$

So, total flux across the cube is 3/4.
(outward)

We can also use flux formula.

Lec-37 Divergence theorem for regions bounded by two surfaces



Imagine two spheres, one inside & one ~~out~~ outside.
We are talking about volume b/w two spheres 1 & 2.

The outward normal for S_2 points outward (away from origin) & outward normal for S_1 points toward origin.

we are talking about region b/w S_1 & S_2.

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma$$

$S \to S = S_1 \cup S_2$.

$$= \iiint_R \nabla\cdot\vec{F}\,dV.$$

If we write n_1 & n_2.

$$-\iint_{S_1} \vec{F}\cdot\vec{n_1}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n_2}\,d\sigma = \iiint_R \nabla\cdot\vec{F}\,dV$$

If $\nabla\cdot\vec{F} = 0 \Rightarrow$

$$\iint_{S_2} \vec{F}\cdot\vec{n_2}\,d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n_1}\,d\sigma$$

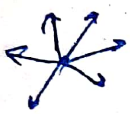
If our divergence is 0, we can replace with any other surface that will be easier to compute.

Lec-38 Deriving Gauss's Law for electric flux via Divergence Theorem

If $\nabla\cdot\vec{F} = 0 \Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}_2\, d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}_1\, d\sigma.$

Example : $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}.$

$$r = \sqrt{x^2+y^2+z^2}$$



$$\nabla\cdot\vec{F} \neq 0$$

$$\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = \frac{3}{r}$$

$$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad \frac{\partial r}{\partial x} = \frac{x}{r};$$

$$2r\frac{\partial r}{\partial y} = 2y \qquad \frac{\partial r}{\partial y} = y/r$$

$$2r\frac{\partial r}{\partial z} = 2z \qquad \frac{\partial r}{\partial z} = z/r$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1\cdot r^3 - x\cdot 3r^2\,\partial r/\partial x}{(r^3)^2}$$

$$= \frac{r^3 - x\cdot 3r^2 \times x/r}{r^6}$$

$$= \frac{r^3 - 3x^2 r}{r^6} = \frac{r^2 - 3x^2}{r^5}$$

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$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^2 - 3x^2}{r^5} \quad -①$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{r^2 - 3y^2}{r^5} \quad -②$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{r^2 - 3z^2}{r^5} \quad -③$$

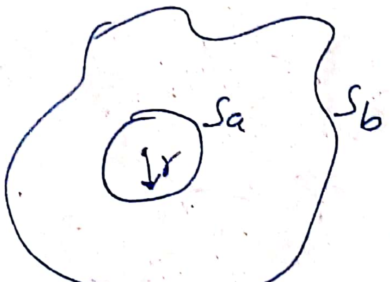
$$\nabla \cdot \vec{F} = ① + ② + ③$$

$$= \frac{3r^2 - 3(x^2+y^2+z^2)}{r^5} = 0.$$

So, $\nabla \cdot \vec{F} = 0$

So, flux across one surface is same as flux across other surface.

So, we compute for nice surface S_a, then the flux will be same for surface S_b.



S_a: sphere of radius a centered at origin.

$$\iint_{S_a} \vec{F} \cdot \vec{n} \, d\sigma \qquad \vec{n} = \frac{\nabla g}{|\nabla g|}$$

$$g : x^2 + y^2 + z^2 = a^2$$

$$\vec{n} = \frac{\langle 2x, 2y, 2z \rangle}{2\sqrt{x^2+y^2+z^2}} = \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{F} = \frac{\langle x, y, z \rangle}{r^3}$$

$r = a$

$$\iint_{S_a} \frac{\langle x, y, z \rangle}{r^3} \cdot \frac{\langle x, y, z \rangle}{r} \, d\sigma$$

$$\iint_{S_a} \frac{r^2}{r^4} \, d\sigma = \frac{1}{a^2} \iint_{S_a} d\sigma$$

$$= \frac{1}{r^2} \iint_{S_a} d\sigma = \frac{4\pi r^2}{r^2} = 4\pi.$$

our r is the radius of sphere $\Rightarrow a$.

$$\vec{F} = \frac{\langle x, y, z \rangle}{a^3}$$

$$\iint_{S_a} \frac{\langle x, y, z \rangle}{a^3} \cdot \frac{\langle x, y, z \rangle}{a} \, d\sigma$$

$$\iint \frac{a^2}{a^4} \, d\sigma = \frac{1}{a^2} \iint d\sigma$$

$$= \frac{1}{a^2} \times 4\pi a^2 = 4\pi.$$

So, flux of $\vec{F}$ across a small sphere $= 4\pi$.
Now flux across S_b is also equal to 4π.

Gauss Law derivation

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \vec{F}$$

$$\iint_S \vec{E}\cdot\vec{n}\,d\sigma \quad \text{(outward flux)}$$

$$= \iint_{S_a} \vec{E}\cdot\vec{n}\,d\sigma$$

sphere of radius a

$$= \frac{q}{4\pi\varepsilon_0} \iint \vec{F}\cdot\vec{n}\,d\sigma$$

$$= \frac{q}{4\pi\varepsilon_0} \times 4\pi = q/\varepsilon_0$$

So flux across an arbitrary surface of an electric field $= q/\varepsilon_0$.

This is gauss's law.

Lec-39 Unified view of Vector calculus

Local properties of fields:

curl $\vec{F} = \nabla \times \vec{F}$

div $\vec{F} = \nabla \cdot \vec{F}$

(at any single point in field we can calculate curl & divergence)

Green's Theorem

We get global property by adding up local property across the region.

Green's Theorem (Divergence form)

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA$$

Divergence Theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_D \nabla\cdot\vec{F}\,dV$$

Flux

---

for a flow/circulation:-

Green's Theorem (circulation form):

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot\hat{k}\,dA$$

Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \cancel{\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma}$$

$$= \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$

Fundamental theorem of Line Integrals

For continuous $\vec{F} = \nabla f$

$$\int_C \vec{F}\,dr = f(B) - f(A)$$

Here ∇f is local, so when we integrate ∇f across the whole boundary we just get result on just boundary end points, in this case $f(B)$ and $f(A)$.

Fundamental Theorem of calculus

If $f(x)$ is differentiable on $[a,b]$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Unifying principle

Integrating a differential operator acting on a field over a domain is the same as adding the field components along the boundary.