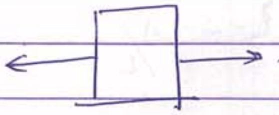
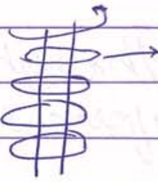
Solid Mechanics

L1



(Normal stress)
Tension (Tensile stress) (Negative is compression)

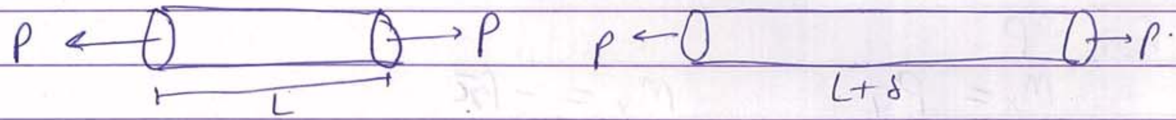
shear stress



Bearing stress (Grabbing by hand)

Truss have only normal stress.
Large columns on airport have only normal stress.

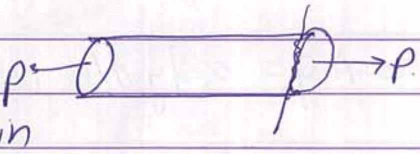
Prismatic Bar under Axial Load



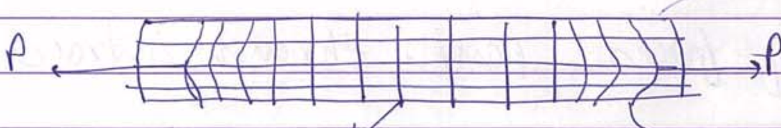
For prismatic bar, area remains same.

$$\sigma = \frac{\text{Axial Load}}{\text{Cross-section Area}} = P/A$$

We can isolate section still load will be P as it is in equilibrium.



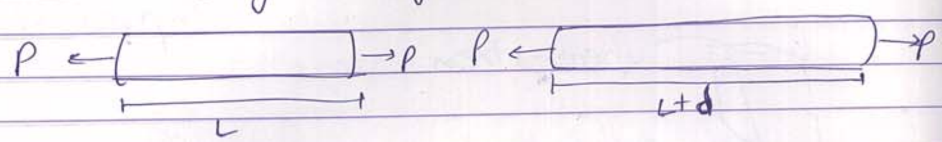
It has some assumption. → St. Venant's principle



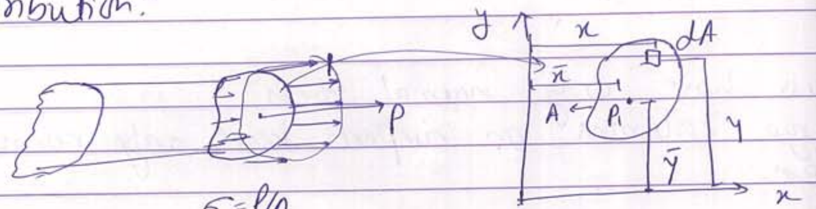
Uniform Deformation.
non-uniform deformation

Normal strain: Prismatic bar under Axial Load

$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{L+\delta-L}{L} = \delta/L$$



* Line of action of forces must pass through the centroid of cross section for uniform stress distribution.



$\sigma = P/A$

Prismatic bar

For P.

$$M_x = P\bar{y} \qquad M_y = -P\bar{x}$$

For σ

$$M_x = \int \sigma \, dA \, y \qquad M_y = \int -\sigma \, dA \, x$$

So, $P\bar{y} = \int \sigma y \, dA$, $P\bar{x} = \int \sigma x \, dA$

$\Rightarrow \sigma A \bar{y} = \sigma \int y \, dA$, $\sigma A \bar{x} = \sigma \int x \, dA$

$$\bar{y} = \frac{\int y \, dA}{A} \qquad \bar{x} = \frac{\int x \, dA}{x}$$

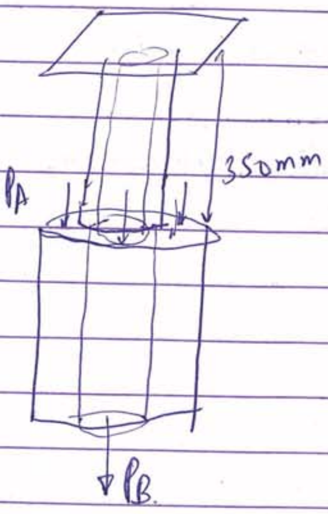
So, line of action of forces, passes through centroid

L2. Q) Two hollow cylinder pipes connected with cap plate.

Upper pipe: $L_1 : 350mm$, $d_1 = 51mm$ $d_2 = 60mm$

Lower pipe: $L_2 = 400mm$, $d_3 = 57mm$ $d_4 = 63mm$

Load $P_A = 7800N$ on top & load P_B applied at bottom



1) a) find P_B, so that tensile in upper pipe is $14.5 MPa$.

2) b) for this case, what is the stress σ in lower pipe?

$$\Rightarrow \quad \frac{P_A + P_B}{A_u} = 14.5 \text{ MPa}$$

$$\frac{7800 + P_B}{\frac{\pi}{4}(60^2 - 51^2)} = 14.5 \text{ MPa}$$

$$P_B = 14.5 \times \left(\frac{\pi}{4}(60^2 - 51^2)\right) - 7800.$$

$$= 3576.88 \text{ N}.$$

For, this, σ in lower $= \frac{3576.88}{A_L} = P_B / A_L$

$$= \frac{3576.88}{\frac{\pi}{4}(63^2 - 57^2)} \text{ MPa} = 6.325 \text{ MPa}.$$

2) a) If P_A is unchanged, find new P_B so that σ is same for upper and lower pipe.

b) for these loads, find tensile ~~stress~~ strains ε in upper and lower pipe if upper elongates by 3.56 mm & downward displacement of lower pipe is 7.63mm

a). σ is same $\Rightarrow \frac{P_A + P_B}{A_u} = \frac{P_B}{A_L}$

$$\frac{7800 + P_B}{\frac{\pi}{4}(60^2 - 51^2)} = \frac{P_B}{\frac{\pi}{4}(63^2 - 57^2)} \Rightarrow 7800 + P_B = \frac{999}{720} P_B$$

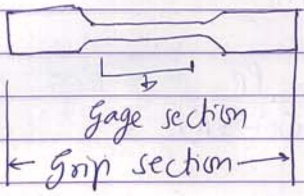
$$P_B = 20129.0 \text{ N} = 20.13 \text{ kN}$$

b) $\varepsilon_u = \dfrac{3.56}{350} = 0.01017$

$\varepsilon_L = \dfrac{7.63 - 3.56}{400} = 0.010175.$

L3 For tensile / compressive testing we have a machine and a specimen.
The ~~machine~~ setup has load cell, extensometer, specimen, Grips, and moving crosshead.

Our specimen looks as →



The elastometer measures elongation of central part of specimen - element strain.

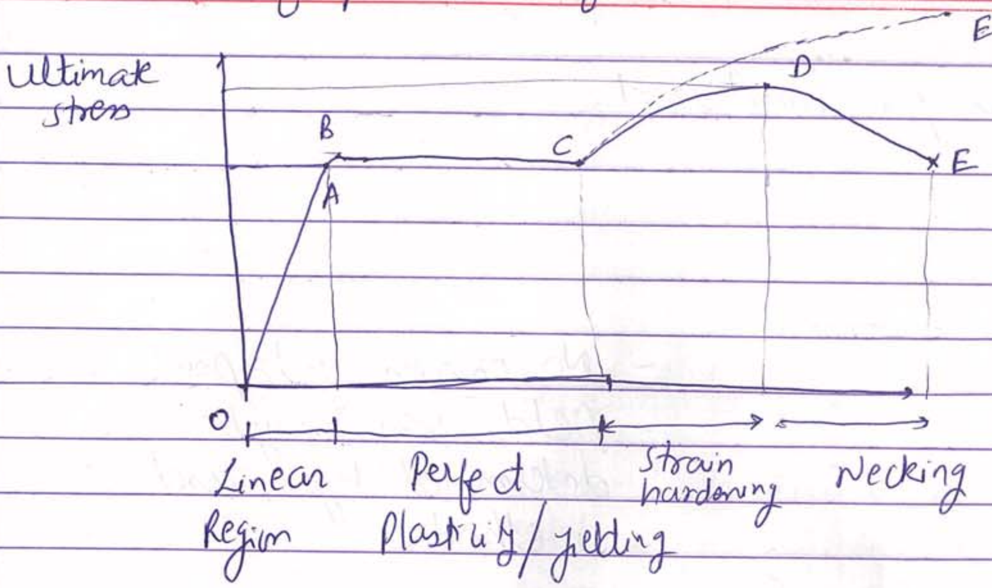
~~Lod~~ Load cell measues force applied to specimen ends → element stress.

At the middle stress is calculated generally.
Load is slowly increased and the elongation is measured.

We get region of necking in our specimen. and then the fracture in that region.

The entire load path, from where we have just started pulling the coupon till the point of fracture, if we keep on slowly measuring the stresses and strains, we get stress strain curve.

→ Elastic point is the maximum stress before permanent deformation starts
→ The yield point is the stress at which a substantial, measurable amount of permanent deformation occurs.



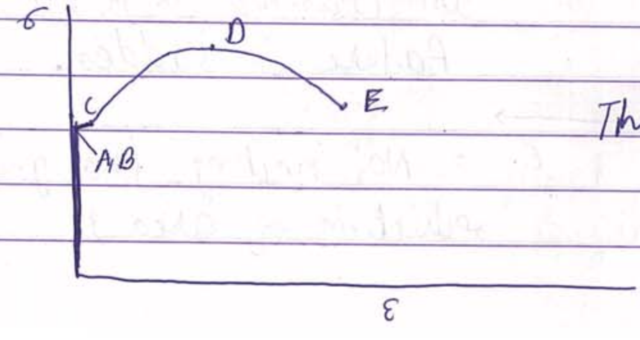
A → Proportional Limit B → Yield stress.

One is engineering (Nominal) stress-strain curve. In this the area remains constant.
Other is true stress-strain curve. In this the ~~area~~ changed area is considered for stress calculation
Till E' is true stress-strain curve
Till E is engineering stress-strain curve. (Nominal stress -strain curve)

The graph is not to scale.

Nominal → Initial area of specimen.
True → actual area of specimen used at the cross-section where failure occurs.

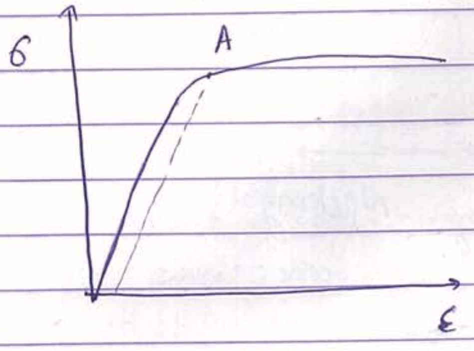
We get this type of curve because steel is ductile material.
(Ductile → Ability to undergo large permanent strains or deformation before failure). (good for accidental designs)



This graph is to scale

The above was for mild steel.

For Al Alloy



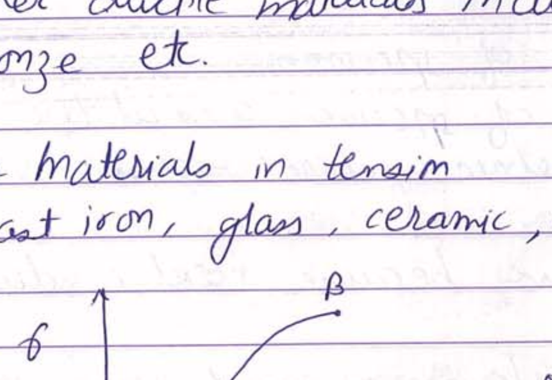
- No obvious yield point
- Yield stress maybe determined by offset method.

Offset method →
- At 0.2% strain (0.002) draw line parallel to linear part. It cuts stress-strain diagram at A, which is defined as yield stress.

- Note Al Alloy is also ductile because it exhibits plasticity (large permanent deformation) before failure.

- Other ductile materials include copper, nickel, bronze etc.

Brittle materials in tension
(cast iron, glass, ceramic, concrete etc.)

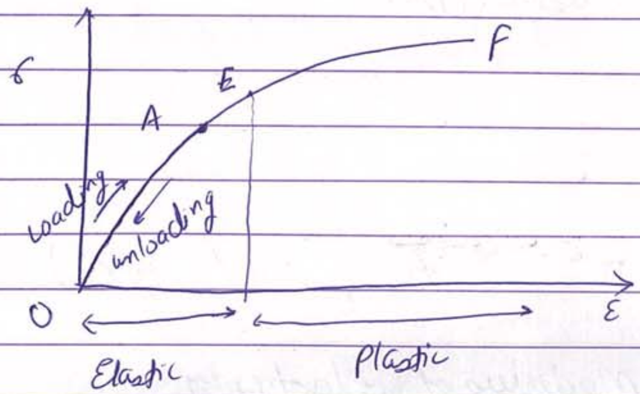


- Materials that fail in tension at relatively low strains are classified as brittle. Failure is sudden.
- No necking (no gradual reduction of area)

- Brittle materials fail only after a little elongation after the proportional limit (point A) is exceeded and doesn't exhibit significant plasticity as ductile materials.

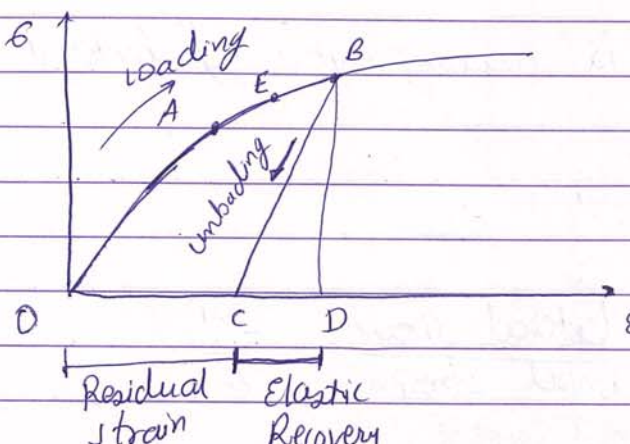
Elasticity, Plasticity & Reloading

The material can be non-linear and elastic also. Till now, we saw material having stress strain curve as linear in elastic region. It can also be



Loading & unloading till point A.

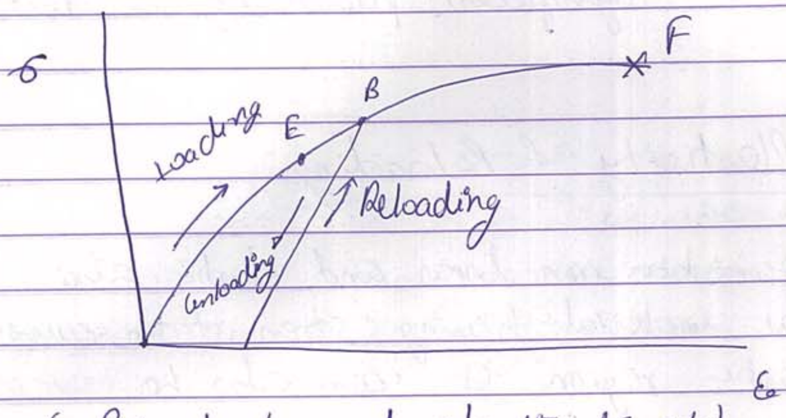
The path followed till A, is not linear. E is the elasticity limit.
If we stretch beyond E, it does not follow back to the same path.



Streatched beyond elastic limit till point B.

After stretching to a certain point, if we unload, it does not goes back to same length. So, there is residual strain.

Now, after residual strain if we stretch again, it goes beyond that new point.



(Reload beyond elastic limit)

Linear Elasticity

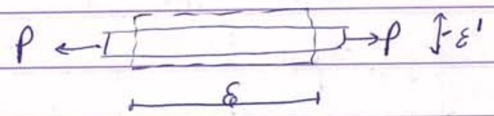
$\sigma = E\varepsilon$ (In curve O-A of mild steel)

E = Youngs modulus / Modulus of elasticity

Steel - 210 GPa
Al - 73 GPa.

Poisson's Ratio

- Axial elongation is accompanied by lateral contraction.

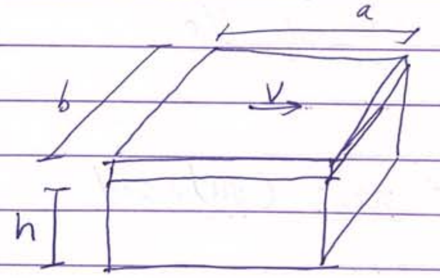


$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = \frac{-\varepsilon'}{\varepsilon}.$$

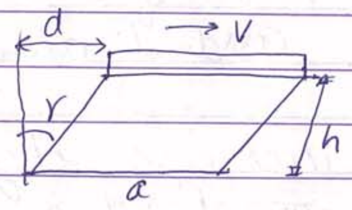
L4 Shear Stress & Bearing stress

Shear stress are stress that act along the surface of an element.
Normal stress act $\perp^{er}$ to the face of element. shear acts along the face of element.

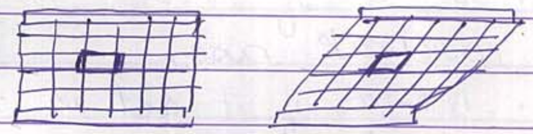
Bridge Bearing pads are used below the bridge, It is not used for ~~no~~ normal forcing coming due to bridge or vehicles, but it is for earthquake, or expansion or compression of bridges. due to temperature stresses.

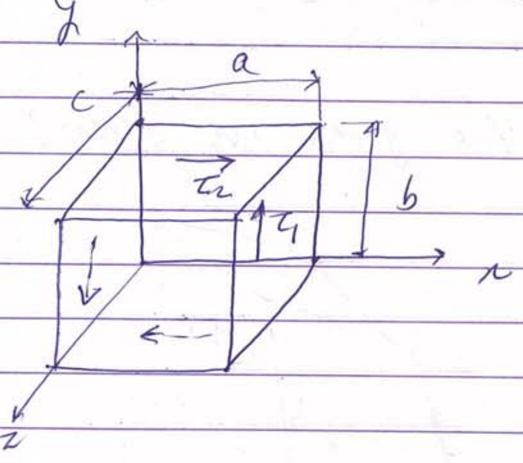


$$\tau_{avg} = \frac{V}{a \times b}$$



Our elements look as →





We only applied τ_2, where did other forces come from.

From eq^m, the bottom force ~~has~~ is also τ_2. (stress)

Now, why do we get τ_1? The τ_2 creates a moment So, τ_1 originates to balance this couple.

$$\tau_2 \times (a \times c) \times b = \tau_1 \times (b \times c) \times a$$

$$\Rightarrow \underline{\underline{\tau_2 = \tau_1 = \tau}}$$

- Shear stresses on opposite (and parallel) faces of an element are equal in magnitude and opposite in direction.

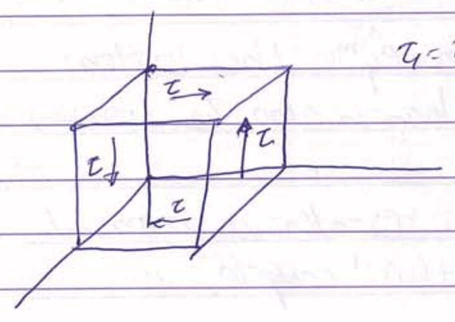
A Sign convention.

- A shear stress acting on a positive face (outward normal) of an element ~~is positive if it~~
  - ~~it acts in +ve dirn of one of the coordinate axes~~
  - is +ve if it acts in +ve dirn of one of the coordinate axes.
  - is -ve if it acts in -ve dirn of one of the coordinate axes.

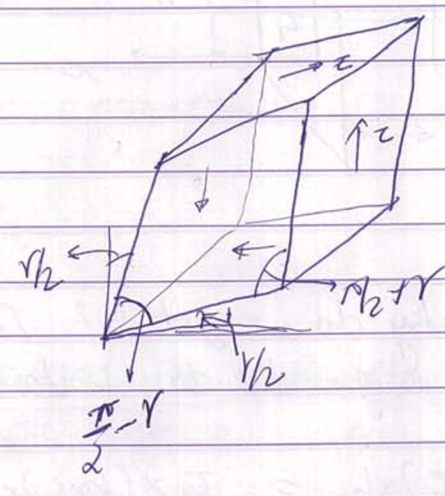
- A shear stress acting on a negative face (outward normal) of an element
  - is +ve if it acts in -ve dirn of one of the coordinate axes
  - is -ve if it acts in +ve dirn of one of the coordinate axes

(negative dirn & negative face gives +ve stress)

Shear Strain



$\tau_1 = \tau_2 = \tau$



γ = shear strain (Measured in degree or radians)

Sign convention

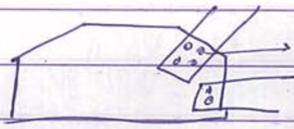
Shear strain in an element is +ve when the angle b/w two +ve faces (or two negative faces) is reduced. The strain is -ve when the angle b/w two +ve (or -ve) faces is increased.

$$\boxed{\tau = G\gamma}$$

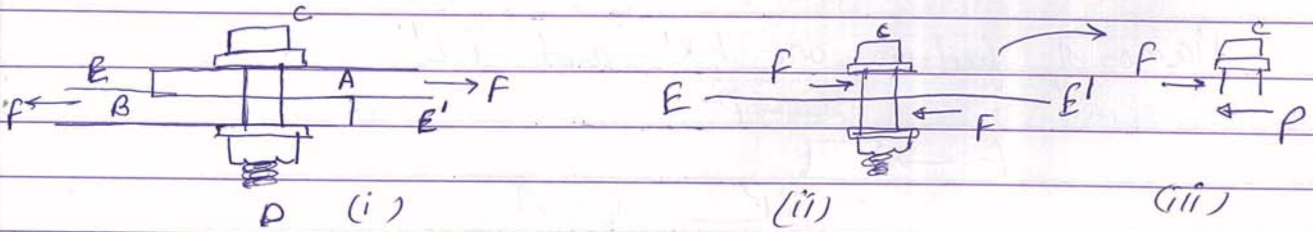
G = shear modulus of elasticity / Modulus of Rigidity.

$$\boxed{G = \frac{E}{2(1+\nu)}}$$ ν = poisson's ratio.

Ex of shear stress :- In pin joint in truss at gusset plate we have shear stress.

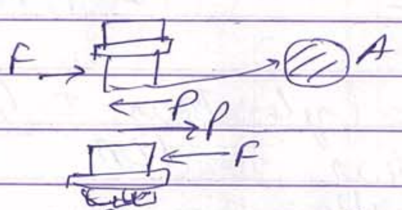


· Bolt in single shear



The bolt will fail at line E - E'.
With eqm we get figure (iii)

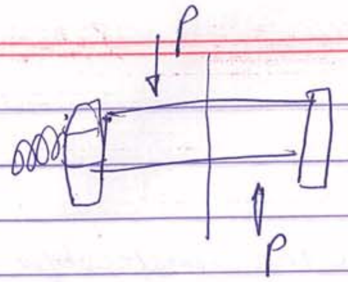
Our P = F;



$P = F$

$$\tau_{avg} = \frac{P}{A} = \frac{F}{A}$$

The bolt has one plane of failure so it is called bolt in single shear.



→ Bolt in single shear.

- Bolt in double shear

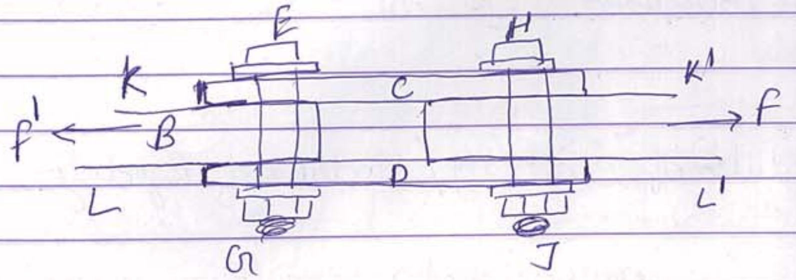
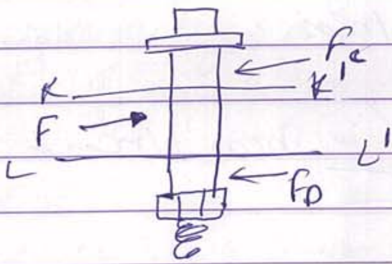


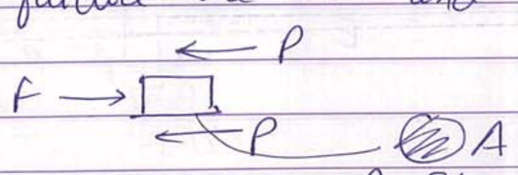
Plate C, D

We see FBD of bolt HJ



For plate C we have F_C & for plate D we have F_D.

Planes of failure are KK' and LL'.



$$F = 2P \quad , \quad P = F/2$$

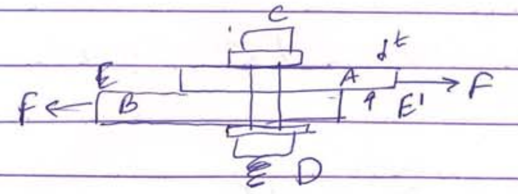
$$F_C = F_D = F/2 , \qquad \tau = P/A = \frac{F}{2A}$$

So, for bolt in single shear $\tau = F/A$ & for bolt in double shear $\tau = F/2A$.

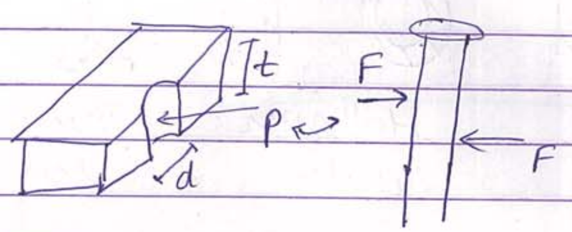
So, bolt in double shear are subjected to half the stresses. So, it is always better to have bolts in double shear.

Bearing stress (Contact stress)

Consider bolt in single shear



If we focus on a particular plate A.



$P = F$.

The bolt applies equal and opposite force to plate. So, over the entire curve surface, a force P is acting. The contact is happening over the entire curved surface. (bearing stress)

Ideally to calculate the stress, we have to take total contact area, but to make thing simpler we take the projected area. (Here, it is $d \times t$)

So, $\sigma_b = \dfrac{P}{d \times t}$ = Bearing stress / Contact stress

Factor of Safety

FS = Factor of safety (n)

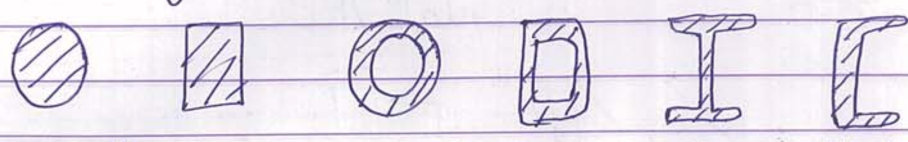
$$n = \frac{\text{ultimate stress}}{\text{allowable stress}} = \frac{\sigma_{ult}}{\sigma_{allw}}$$

Considerations -
- Uncertainity in material properties, loadings, no. of loading cycles, types of failure
- Maintenance requirements & deterioration effects
- Risk to life & property

3.1 Axial Loading (Uniform) (L5)

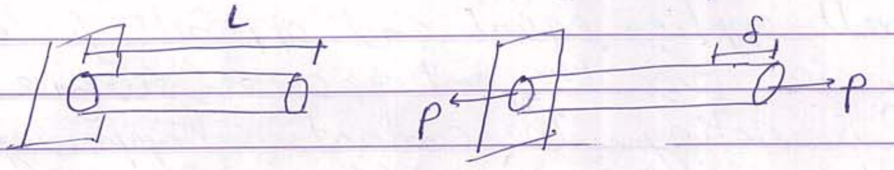
The load acting along the axis of structure is called axial load~~ing~~

We will first see prismatic bars. (cross-section remains same along length)



Solid — Hollow or Tubular — Thin walled open cross section.

Our derivation will be similar for all prismatic bars.



$\sigma = \frac{P}{A}$, $\varepsilon = \frac{\delta}{L}$, and in linear region $\sigma = E\varepsilon$

$$\frac{P}{A} = E\frac{\delta}{L}$$

$$\boxed{\delta = \frac{PL}{EA}}$$

AE / EA = Axial Rigidity.

As A and E increase, it make the structure more rigid, so deflection becomes lesser and lesser.

$$\delta = \left(\frac{L}{AE}\right) \times P \Rightarrow \delta = P \times f$$

* deformation under unit load also known as flexibility (f)

$$P = \left(\frac{AE}{L}\right) \times \delta \qquad \begin{bmatrix} AE/L \rightarrow \text{Stiffness} \\ L/AE \rightarrow \text{Flexibility} \end{bmatrix}$$

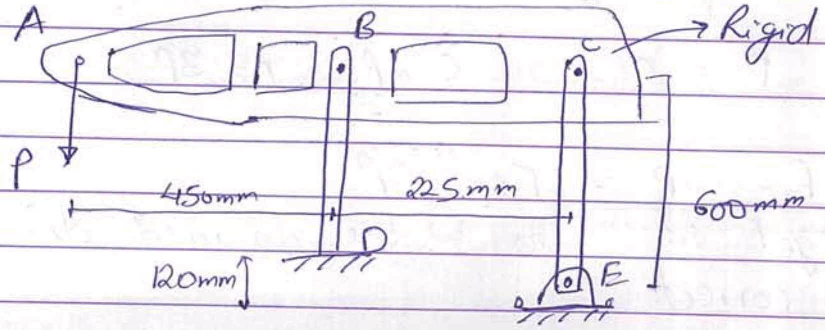
$F = kx$
k: stiffness

$\frac{AE}{L}$ → Load for unit deflection (Known as stiffness 'K')

$\boxed{K = 1/f}$ (Stiffness is opposite of flexibility)

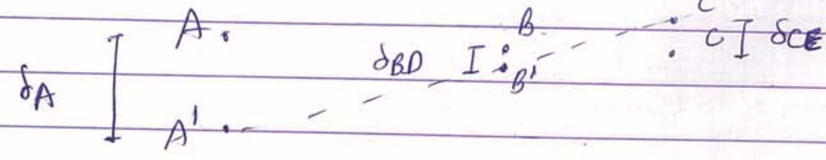
3.2 Axial Loading (Uniform) - Problem (2.6)

(Q) Find δ_{BD}, δ_{CE}, vertical deflection of A



$P = 23.2\,kN$, $A_{BD} = 1020\,mm^2$, $A_{CE} = 520\,mm^2$
$E = 205\,GPa$.

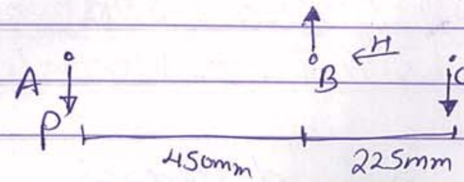
Let us draw deformed configuration



According to diagram BD must be in compression & CE must be in tension.

$$\delta_{BD} = \frac{F_{BD}\,L_{BD}}{A_{BD}\,E_{BD}}$$

As, we have considered compression in BD so force should act upward in frame ABC at B location. And conversely in location C.



$P = 23.2$ kN

The horizontal force has to be 0.

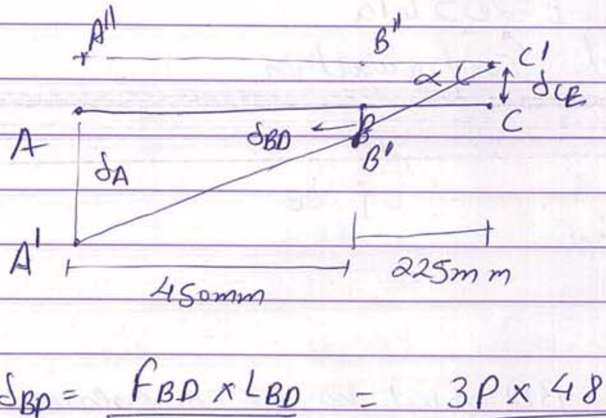
$$\Sigma F_x = 0 \quad H = 0$$
$$\Sigma F_y = 0, \quad \Sigma M_A = 0$$

$$P + C = B$$
$$B \times 450 = C \times 675$$
$$B = \frac{3}{2} C$$
$$P = \frac{C}{2} \Rightarrow C = 2P, \quad B = 3P$$

$$F_{CE} = 2P, \quad F_{BD} = 3P$$

As we get +ve, so the assumed directions are correct.

Now, we can calculate deflections.



$$\delta_{BD} = \frac{F_{BD} \times L_{BD}}{A_{BD} \, E_{steel}} = \frac{3P \times 480}{1020 \times 205} = 0.16 \text{ mm}$$
(compression)

$$\delta_{CE} = \frac{F_{CE} \times L_{CE}}{A_{CE} \times E_{steel}} = \frac{2P \times 600}{520 \times 205} = 0.26 \text{ mm.}$$
(expansion)

By similar Δ $\quad \frac{A'A''}{A''C'} = \frac{B'B''}{B''C'}$

$$\frac{\delta_A + \delta_{CE}}{450+225} = \frac{\delta_{BD}+\delta_{CE}}{225}$$

$$\delta_A + \delta_{CE} = \frac{3}{2}(\delta_{BD}+\delta_{CE})$$

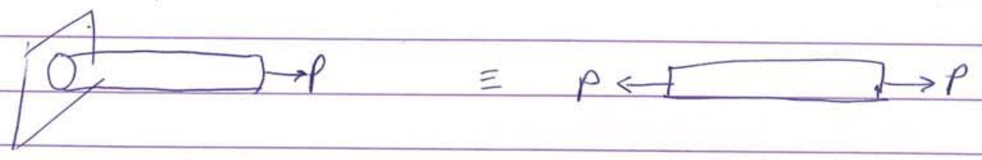
$$\delta_A = \frac{3}{2}\delta_{BD} + \frac{\delta_{CE}}{2}$$

$$= \cancel{\frac{3}{2}\times 0.16 + \frac{0.26}{2} = 0.24 + 0.13 = 0.37\,mm}$$

$$\delta_A = 3\delta_{BD} + 2\delta_{CE} = 3\times 0.16 + 2\times 0.26$$

$$= 1.00\,mm$$

(L7) 3.2 Axial Loading (Non-uniform)



If we chop it off onto smaller sections we get

$P \leftarrow \square \rightarrow P \quad P \leftarrow \square \rightarrow P \quad P \leftarrow \square \rightarrow P \quad P \leftarrow \square \rightarrow P$

dx

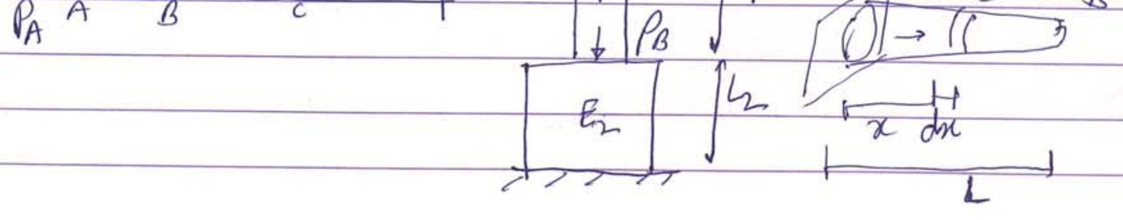
So, every part is in equilibrium
(individual section)

We know, $\delta = \frac{PL}{AE}$

For small section $\boxed{d\delta = \frac{P\,dx}{AE}}$

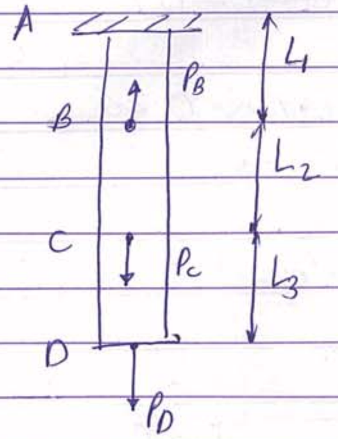
As we see P on entire P, this is uniform loading.

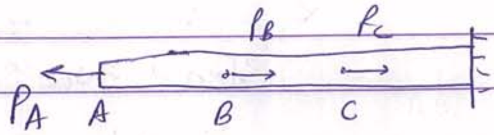
Let us see for ~~an~~ non-uniform loads



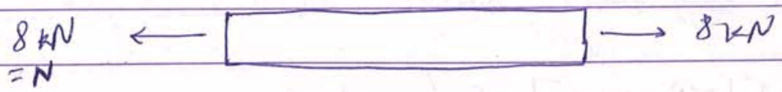
In the above 3 cases, load is changing in ①,
E, A are changing in ②. Area is changing in ③ along with forces.
We will look at 3 cases individually.

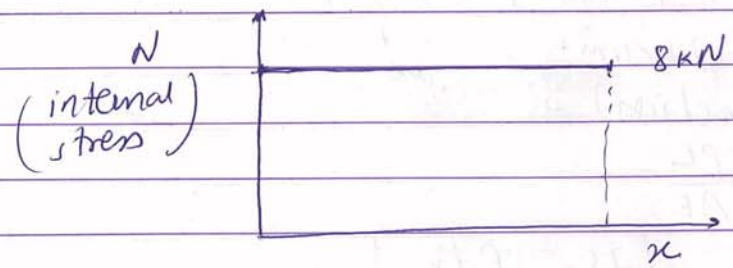
Ⓐ Bars with intermediat axial load
(Cross section area is constant)





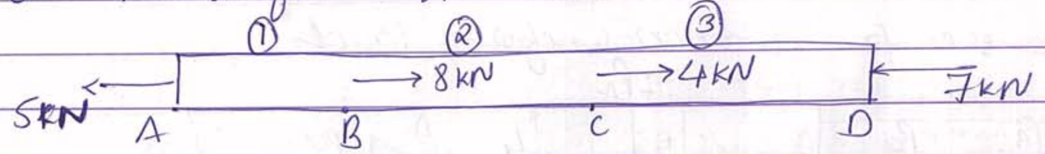
For uniform axial load





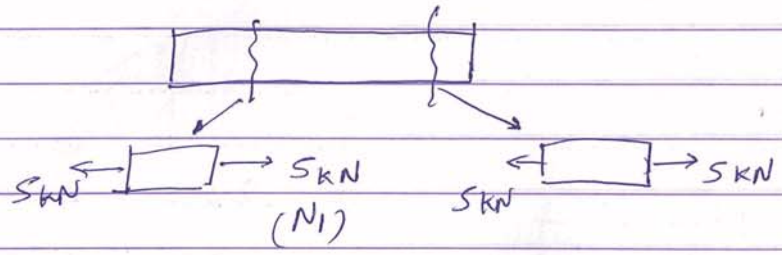
(Convention - Tension is +ve.)

For non-uniform axial load



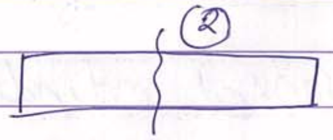
This structure is in equilibrium. (12 kN →, 12 kN ←)
Let us see internal stresses.

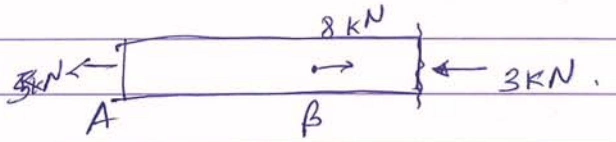
Let us see section ①



$N_1 = 5kN$.

Just to the left of B, we will have 5kN.
At point B, we will have a discontinuity.



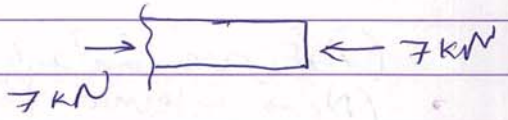


(Internal for $N_2 = 3kN$)

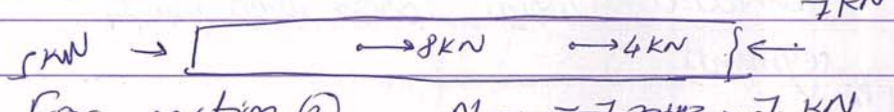
It will be in compression.

In section ②, $N_3 = -3kN$.

For section ③, we can look from right side

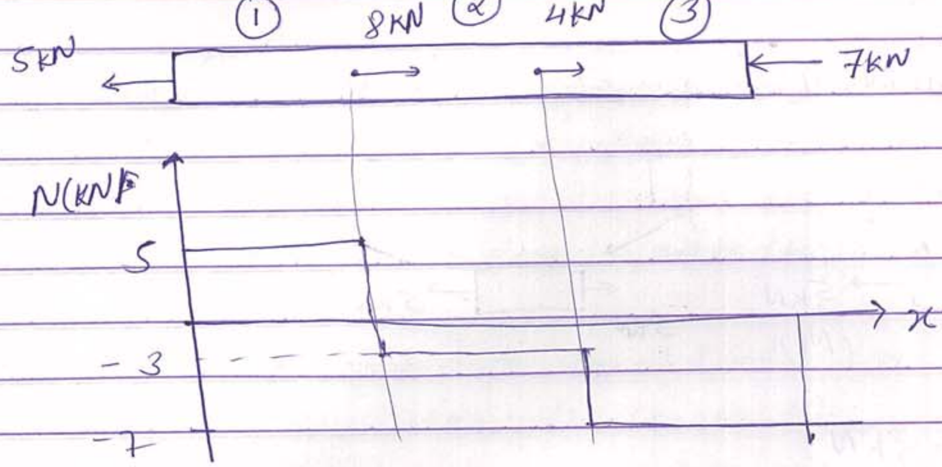


If we look from left



For section ③, $N_3 = \cancel{-783} -7\ kN$.

So, our ~~interf~~ internal forces ~~to~~ vary by where we are.
So, how does total deformation looks like.



N is the internal ~~stress~~ force. It is stress distribution

For total deformation, we look at individual section ①, ② & ③.

for ①, $\delta_1 = \dfrac{5 \times L_{AB}}{AE}$

For ②, $\delta_2 = \dfrac{-3 \times L_{BC}}{AE}$

For ③, $\delta_3 = \dfrac{-7 \times L_{CD}}{AE}$

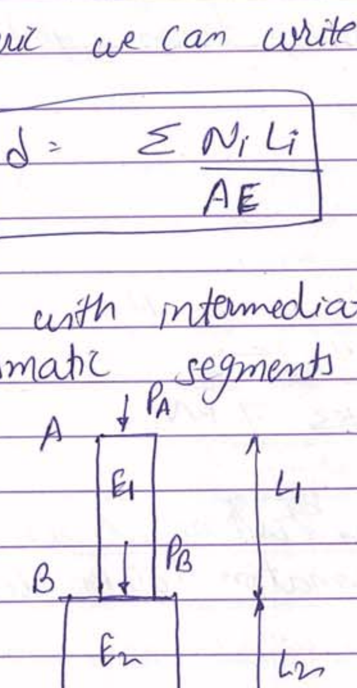
Total deformation $= \delta_1 + \delta_2 + \delta_3$

So, generic we can write as

$$\boxed{\delta = \sum \frac{N_i L_i}{AE}}$$

(A, E are constants)
($N_i \rightarrow$ Internal Forces)

(B) Bars with intermediate axial loads and multiple prismatic segments



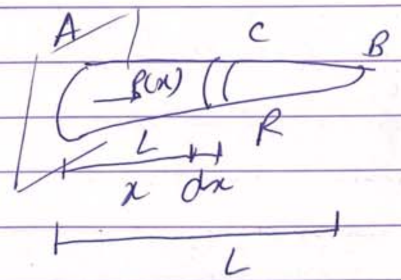
Here A is changing, so, the deflection will be

$$\delta = \sum \frac{N_i L_i}{A_i E}$$

Now, if material also changes, it will be

$$\boxed{\delta = \sum \frac{N_i L_i}{A_i E_i}}$$

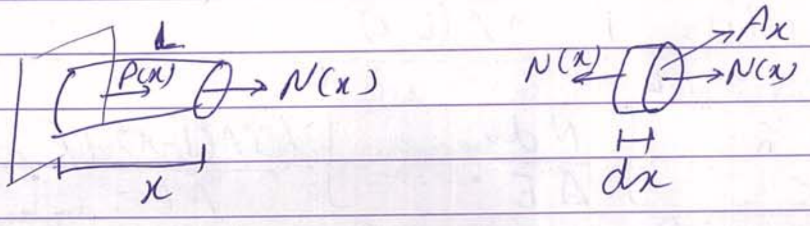
(c) Bars with continuously varying loads or dimension



$\delta = ?$

Bar is assumed weightless, so only axial load is acting.

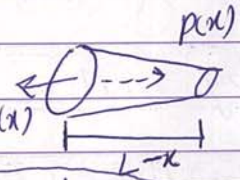
$\equiv$



$$d\delta = \frac{N(x)\,dx}{A(x)\,E}$$

$$\boxed{\delta = \int_0^L \frac{N(x)\,dx}{A(x)\,E}}$$

For right section



~~N(x)=~~

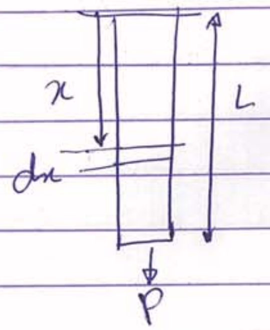
$$\boxed{N(x) = \int_x^L p(x)\,dx}$$

Next we will see examples

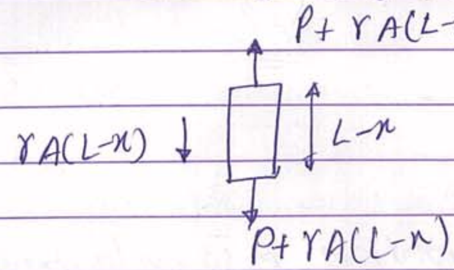
(L8) Axial Loading (non-uniform) part2- problem

Find the end deflection, if the bar is hanging ~~until~~ under its own weight and also supporting a load P

γ = unit weight of material.



For a section, from bottom



So, $N(x) = P + \gamma A(L-x)$

$$\delta = \int_0^L \frac{N\,dx}{AE} = \int_0^L \frac{(\gamma A(L-x) + P)\,dx}{AE}$$

$$= \frac{\left[\gamma A\left(Lx - \frac{x^2}{2}\right) + Px\right]_0^L}{AE}$$

$$\delta = \frac{\gamma L^2}{2E} + \frac{PL}{AE}$$

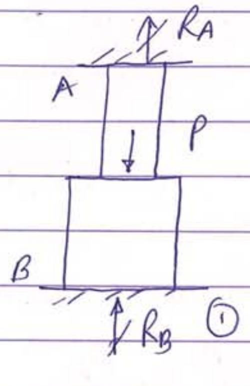
If selfweight is not present, deflection is $\frac{PL}{AE}$,

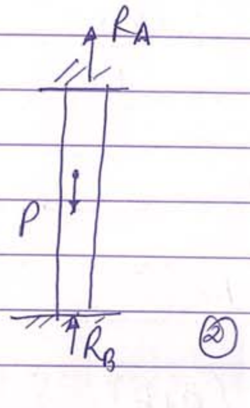
we can also check by units.

(9) Axial Loading - Statically indeterminate cases & Thermal stresses

For statically determinate structure equilibrium equations are sufficient.

For statically indeterminate structure, equilibrium equations are not sufficient.



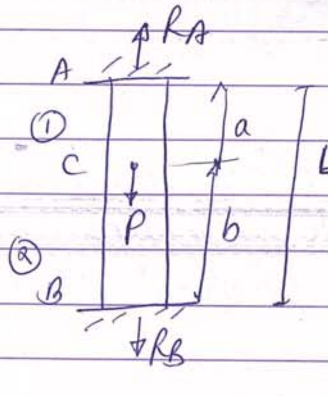


We can't find R_A & R_B using only equilibrium eq^{n}s.
We will use eq^{n}s of compatibility.

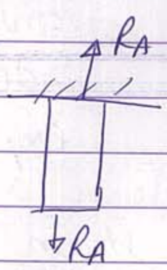
$\boxed{\delta_{AB} = 0}$ for case ① & ②

⇒ Change in length of the bar must be compatible with conditions at the support.

For



For section in ①



$$\delta_{AC} = \frac{R_A \times a}{AE} \quad -①$$

For section ②

$$\delta_{BC} = \frac{R_B \times b}{AE} \quad -②$$

$$R_B + P = R_A \quad -③$$

$$\delta_{AB} = 0 = \delta_{AC} + \delta_{BC} \quad -④$$

$$\delta_{AB} = 0 \qquad \frac{R_A \times a}{AE} + \frac{R_B \times b}{AE} = 0$$

$$R_B = -\frac{R_A \times a}{b}$$

$$R_B + P = R_A$$

$$-\frac{R_A \times a}{b} + P = R_A$$

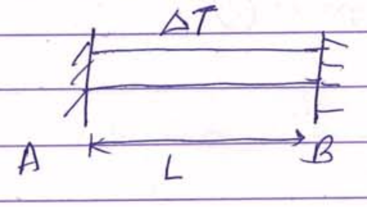
$$P = R_A\left(\frac{a+b}{b}\right) \Rightarrow R_A = \frac{bP}{a+b} = \frac{Pb}{L}$$

$$R_B = -\frac{Pb}{L} \times \frac{a}{b} = \frac{-Pa}{L}$$

So, R_B is compressive & R_A is tensile.

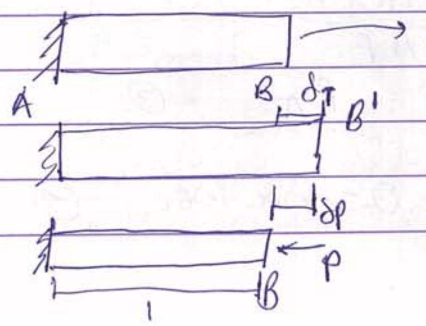
In this way we solve indeterminate structure.

Thermal Stress



- only restrained structures/indeterminate structures will experience thermal stress.

Here don't have strain but stresses.
How do we tackle this problem? The easiest way is to first let the bar expand due to temperature and then to push it back to what its original configuration was.



$\alpha \to$ coeff of thermal expansion

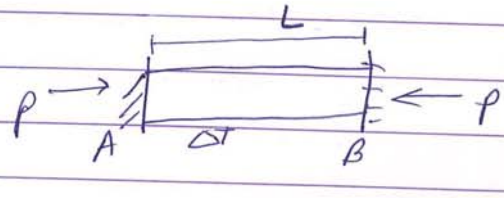
$$\delta_T = L(\Delta T)\alpha$$

$$\delta_P = \delta_T \Rightarrow \boxed{\frac{PL}{AE} = \delta_T} = L\alpha\Delta T$$

$$\Rightarrow \frac{PL}{AE} = L\alpha\Delta T$$

$$\Rightarrow \boxed{P = AE\alpha\Delta T}$$

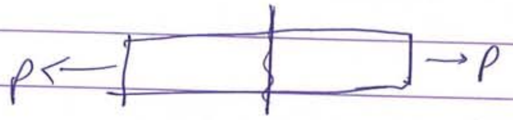
So, in our original structure



$P = AE\alpha\Delta T.$

(L10) Axial stresses on inclined sections

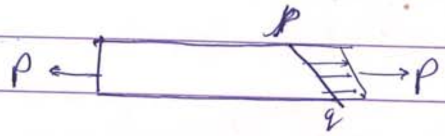
Until now we saw



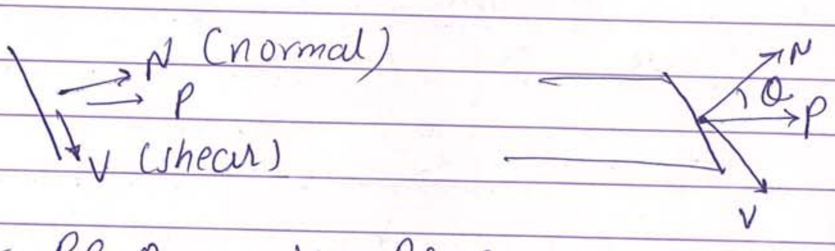
$\sigma_x = P/A$

If we cut ~~t~~ at a particular slope

, what are stresses at a plane

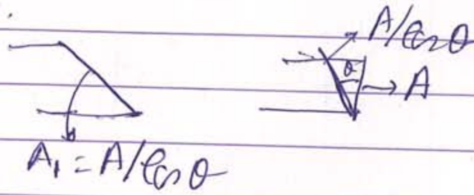


We resolve P into 2 parts



$N = P\cos\theta \qquad V = P\sin\theta.$

Then area also changes.



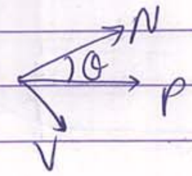
$$\sigma_\theta = \frac{N}{A_1} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta.$$

$$\sigma_\theta = \frac{P}{A}\cos^2\theta \qquad \frac{P}{A} = \sigma_x$$

$$\sigma_\theta = \sigma_x \cos^2\theta. = \boxed{\frac{\sigma_x}{2}(1+\cos 2\theta)}$$

σ_θ acts normal to ~~cross~~ cut cross section.

At $\theta = 0 \qquad \sigma_\theta = \sigma_x$.



For V, we denote τ_θ.

$$\tau_\theta = \frac{V}{A_1} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta$$

$$\boxed{\tau_\theta = \frac{P}{2A}\sin 2\theta}$$

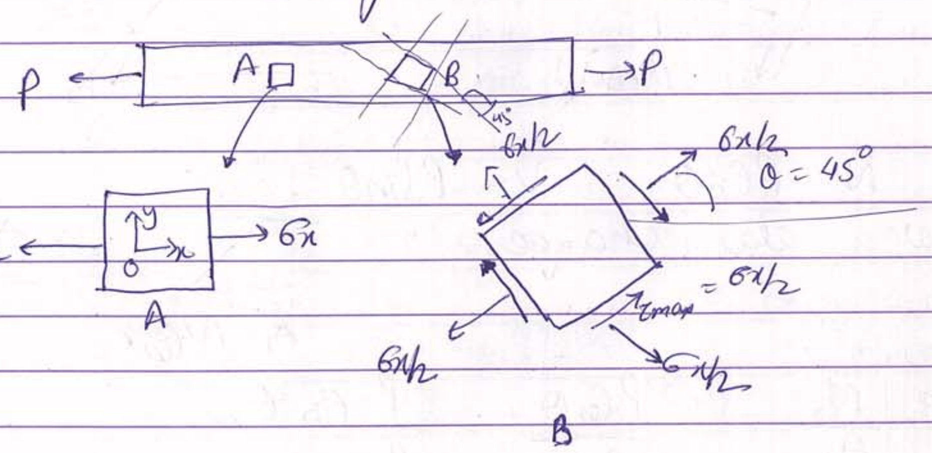
At $\theta = 0$, $\tau_\theta = 0$ at $\theta = \pi/4$ $\tau_\theta = P/2A = \tau_{max}$.

$\tau_{max} = \sigma_x/2$.

So, at $\theta = 0$, $\sigma_{max} = \sigma_x$, $\tau_\theta = 0$

at $\theta = \pi/4$, $\sigma_\theta = \frac{\sigma_x}{2}$, $\tau_\theta = \frac{\sigma_x}{2}$

It happens in the same structure and depends on how we are looking at.



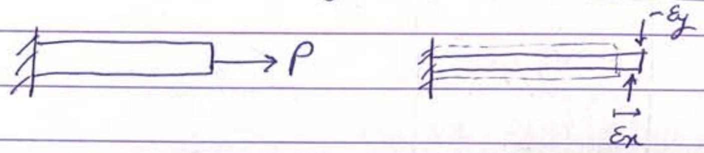
If we have materials that are extremely strong in tension or compression but beyond a particular point they are weak in shear.
So, If a material is weak in shear it will fail at a particular angle.

(eg: Push/compression of ~~wod~~ wood results in failure at 45°)

(Lec 11) Generalized Hooke's law in 3D (only axial stress)

We discussed $\sigma = E\varepsilon$.

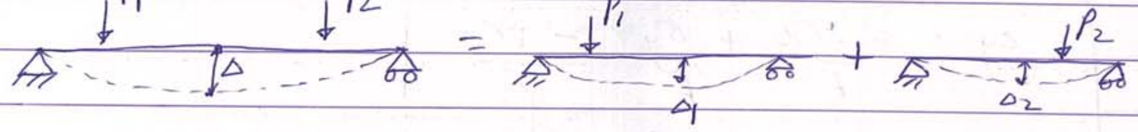
Poisson's ratio ranges from $\boxed{0.0 \text{ to } 0.5}$.



$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = \frac{-\varepsilon'}{\varepsilon} = \frac{-\varepsilon_y}{\varepsilon_x}$$

$$\Rightarrow \varepsilon_y = -\nu\varepsilon_x \quad \text{and} \quad \varepsilon_z = -\nu\varepsilon_x$$

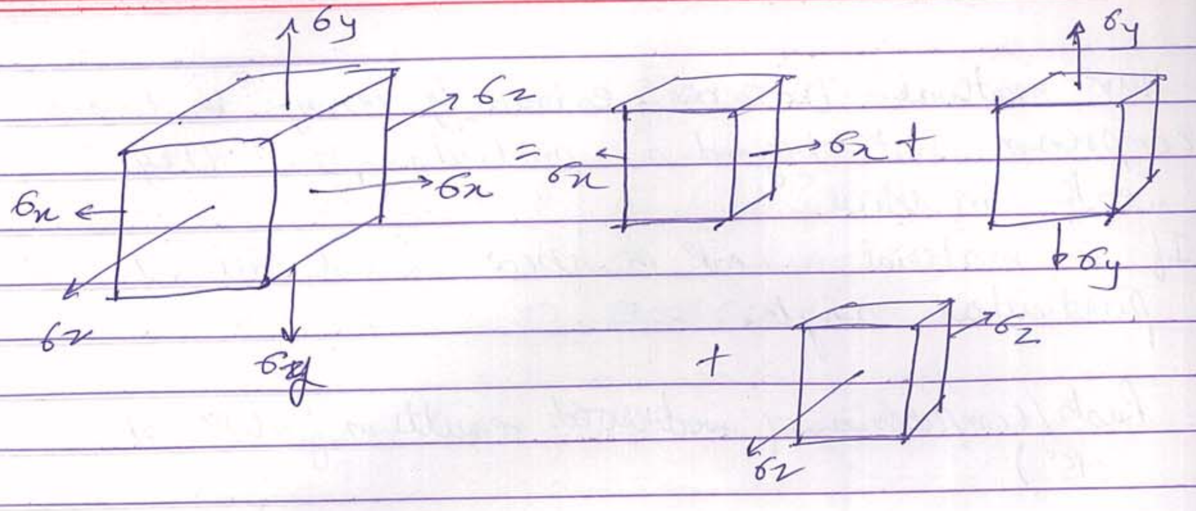
- In case of multiaxial loading (no shear)



Principle of superposition

Effect of combined loading = Combination of individual loading effects

$$\Delta = \Delta_1 + \Delta_2$$



So, for σ_x, we will have strain in y & z.

Due to	ε_x	ε_y	ε_z
σ_x	σ_x/E	$-\nu\varepsilon_x = -\frac{\nu\sigma_x}{E}$	$-\nu\sigma_x/E$
σ_y	$-\nu\sigma_y/E$	σ_y/E	$-\nu\sigma_y/E$
σ_z	$-\nu\sigma_z/E$	$-\frac{\nu\sigma_z}{E}$	$\frac{\sigma_z}{E}$

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \varepsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \varepsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned}}$$

Dilation

Change of volume per unit volume is called dilation.

Change of volume (e) after being stressed:

$$e = \underbrace{(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)}_{\text{final volume}} - \underbrace{1}_{\text{initial volume}}$$

$$= (1+\varepsilon_x+\varepsilon_y+\varepsilon_z) - 1$$

(Neglecting strain products since small)

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Here we have considered original volume as 1
e = change in volume per unit volume = dilation

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

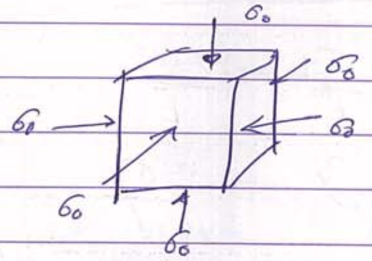
$$= \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} - \frac{v\sigma_z}{E} - \frac{v\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{v\sigma_z}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{\sigma_x+\sigma_y+\sigma_z}{E} - \frac{2v(\sigma_x+\sigma_y+\sigma_z)}{E}$$

$$\boxed{e = \frac{1-2v}{E}(\sigma_x+\sigma_y+\sigma_z)}$$

Now, if we considered hydrostatic condition.
So, hydrostatic pressure is same at every point.

$$\sigma_x = \sigma_y = \sigma_z = -\sigma_0$$



For hydrostatic stress (Copied)

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2v}{E}(-\sigma_0 - \sigma_0 - \sigma_0)$$

$$= \frac{-3\sigma_0(1-2v)}{E}$$

Assuming $K = \dfrac{E}{3(1-2v)}$, $e = -\sigma_0/K$

$$\boxed{K = \frac{E}{3(1-2\nu)}} \rightarrow \text{Bulk Modulus}$$

$$\boxed{e = -\frac{\sigma_0}{K}}$$

The opposite of bulk modulus is compressibility.

$k = 1/K$

$$\text{Compressibility} = \frac{1}{\text{Bulk Modulus}(K)} = k$$

It is also defined as $\boxed{K = -V\frac{dP}{dV}}$

(12) <u>Generalized Hooke's Law (Axial + Shear)</u>

For multiaxial loading (no shear)

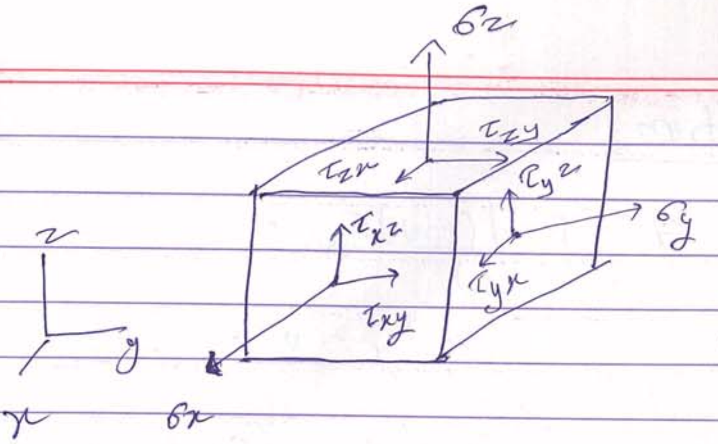
We saw principle of superposition

effect of combined loading = combination of individual loading effects

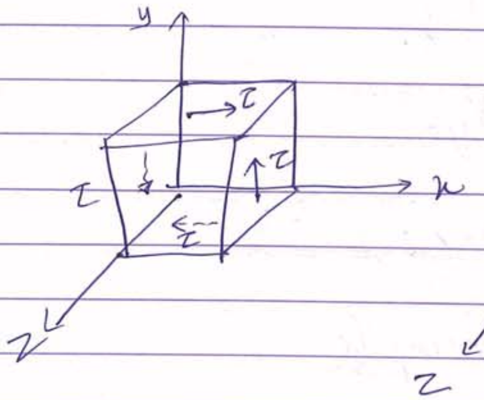
$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \varepsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \varepsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned}}$$

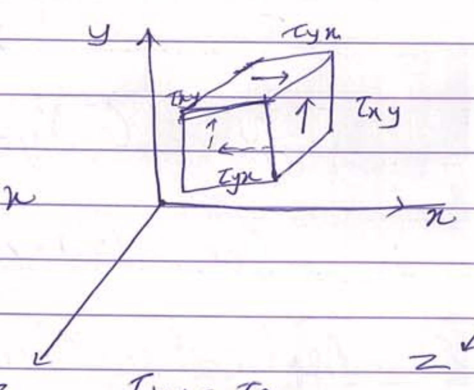
Now, we will see for axial + shear.

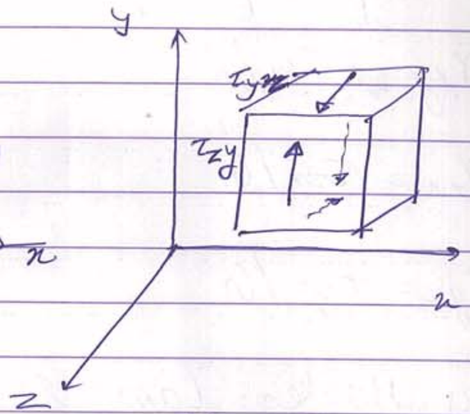
For any point inside a body, the state of stress will be



For shear stress

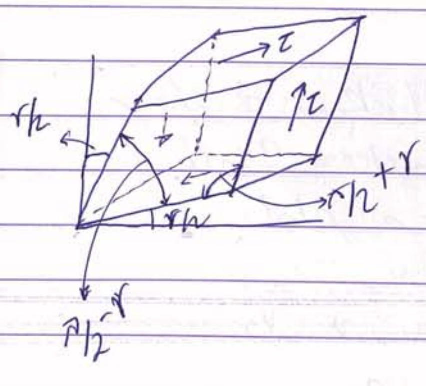






$\tau_{yx} = \tau_{xy}$

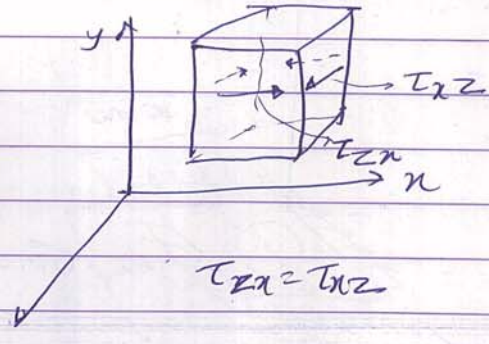
$\tau_{yz} = \tau_{zy}$



$\boxed{\tau_{xy} = G\gamma_{xy}}$

$\boxed{\tau_{yz} = G\gamma_{yz}}$

$\boxed{\tau_{xz} = G\gamma_{xz}}$



$\tau_{zx} = \tau_{xz}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \; ; \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \; ; \quad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

So, for axial + shear we have

$\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu\sigma_y}{E} - \dfrac{\nu\sigma_z}{E}$	$\gamma_{xy} = \dfrac{\tau_{xy}}{G}$
$\varepsilon_y = \dfrac{-\nu\sigma_x}{E} + \dfrac{\sigma_y}{E} - \dfrac{\nu\sigma_z}{E}$	$\gamma_{zx} = \dfrac{\tau_{xz}}{G}$
$\varepsilon_z = \dfrac{-\nu\sigma_x}{E} - \dfrac{\nu\sigma_y}{E} + \dfrac{\sigma_z}{E}$	$\gamma_{yz} = \dfrac{\tau_{yz}}{G}$

Transformation

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = \frac{-\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = \frac{-\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \tau_{xy}/G$$

$$\gamma_{zx} = \tau_{zx}/G$$

$$\gamma_{yz} = \tau_{yz}/G$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y+\varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_x+\varepsilon_z)\right]$$

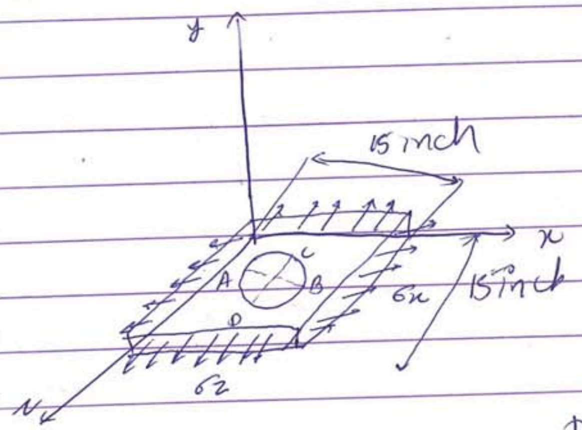
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x+\varepsilon_y)\right]$$

$$\tau_{xy} = G\,\gamma_{xy}$$

$$\tau_{yz} = G\,\gamma_{yz}$$

$$\tau_{zx} = G\,\gamma_{zx}$$

(13) Hooke's Law 3D - Application Example



Al plate of thickness $= t = 3/4$ in.
Circle of diameter $= 9$ inch is scribed on a plate
Stresses $\sigma_x = 12$ ksi, $\sigma_z = 20$ ksi,
$E = 10\times10^6$ psi, $\nu = 1/3$.

Determine change in:
a) Length of diameter AB
b) Length of diameter CD
c) Thickness of plate
d) volume of plate.

δ_{AB}, δ_{CD}, δ_t, ΔV.

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\overset{0}{\sigma_y}}{E} - \frac{\nu\sigma_z}{E} = \frac{\sigma_x}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{12}{E} - \frac{1}{3}\times\frac{20}{E} = \frac{16}{3E}$$

$$\frac{\delta_{AB}}{AB} = \varepsilon_x \Rightarrow \delta_{AB} = 9\text{in}\times\varepsilon_x$$

$\varepsilon_x = \frac{16\ ksi}{3\times10\times10^6\ psi} \Rightarrow \frac{16\times1000}{3\times10^7} \Rightarrow \frac{16}{3}\times10^{-4} = 5.33\times10^{-4}$

$\delta_{AB} = \cancel{9}^{3}\times\frac{16}{\cancel{3}}\times10^{-4}\ in = 48\times10^{-4}\ in = 4.8\times10^{-3}\ in$

Similarly,

$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \overset{\to 0}{\frac{\nu\sigma_y}{E}} + \frac{\sigma_z}{E} = -\frac{1}{3}\times\frac{12}{E} + \frac{20}{E} = \frac{16}{E}$

$\delta_{CD} = CD\times\varepsilon_z = 9\,inch\times\frac{16}{E}$.

Thickness of plate

$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \overset{\to 0}{\frac{\sigma_y}{E}} - \frac{\nu\sigma_z}{E} = -\frac{\nu}{E}(\sigma_x+\sigma_y) = -\frac{1}{3E}(32)$

$\delta_t = t\times\left(\frac{-32}{3E}\right) = 9\times\left(\frac{-32}{3E}\right)$.

Volume of plate.

$e = \text{dilation} = \cancel{\varepsilon_x\,\varepsilon_y\,\varepsilon_z}\ \varepsilon_x+\varepsilon_y+\varepsilon_z = \frac{dV}{V}$. 48 +16 64

$dV = V\times(\varepsilon_x+\varepsilon_y+\varepsilon_z) = V\times\left(\frac{16}{3E}+\frac{16}{E}-\frac{32}{3E}\right)$

$= V\times\left(\frac{32}{3E}\right)$

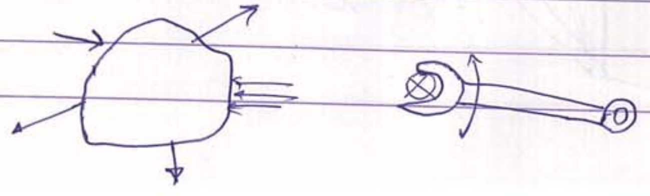
Change in volume $= \frac{32V}{3E}$

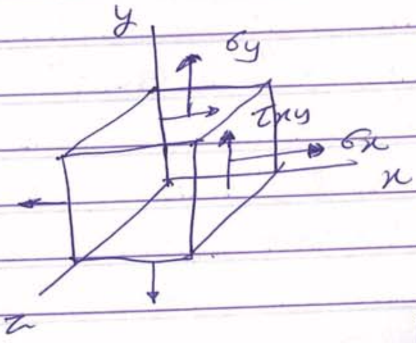
(14) Plane Stress and Plane Strain Problem

1. Plane stress problem.

This is for the problem where stress acting in one dir^n is 0.

eg. Thin members





For plane stress

$\sigma_z = 0$

$\tau_{xz} = 0$

$\tau_{yz} = 0$

i.e. in one dir^n stress is 0.

here no load acting in z-axis.

So for this case, Transformation.

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \sigma_y/E$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

$$\gamma_{xy} = \tau_{xy}/G$$

$\Rightarrow$

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

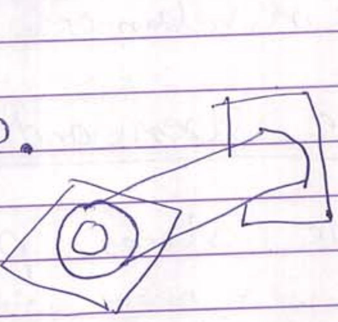
$$\tau_{xy} = G\gamma_{xy}.$$

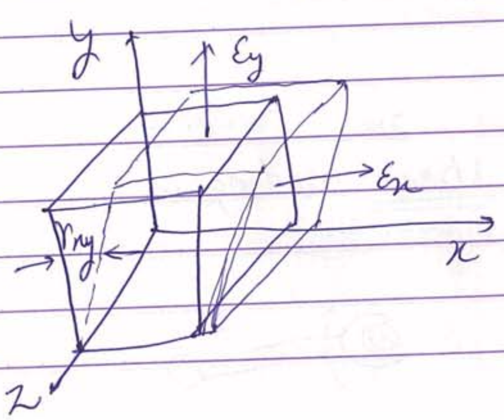
2. Plane strain Problems

This is for long thick members. where strain in one dir^n is 0.

ex: dams, cylindrical pressure tube constrained by rigid walls.

$\varepsilon_z = 0$, $\gamma_{xz} = 0$, $\gamma_{yz} = 0$.





$\varepsilon_z = 0$, $\gamma_{xz} = 0$, $\gamma_{yz} = 0$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right]$$

$$\varepsilon_y = \frac{-\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad \sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\varepsilon_x\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu(\varepsilon_x + \varepsilon_y)\right]$$

$$\tau_{xy} = G\gamma_{xy}$$

Transformation

→ Plane stress and plane strain are 2D simplification in solid mechanics used to reduce 3D analysis to 2D plane for faster, efficient FEA modelling.

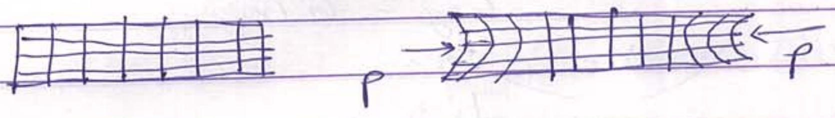
→ Plane stress applies to thin components where out of plane stress is 0.

→ Plane strain applies to long, thick or constrained structures (eg. dams, tunnels) where out of plane strain is ~~≠~~ 0.

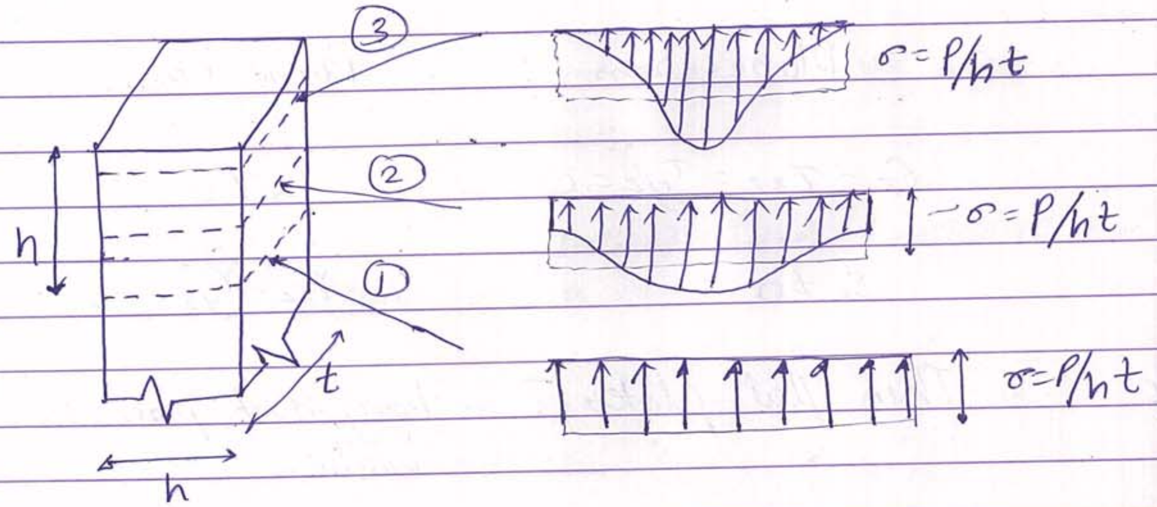
	Plane Stress	Plane Strain
Stress	$\sigma_z = \tau_{xz} = \tau_{yz} = 0$	$\sigma_z \neq 0$
Strain	$\varepsilon_z \neq 0$	$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
Geometry	Thin, flat plates	Long, thick, prismatic bodies
Example	Loaded sheet metal	Dams, tunnel, pipes.

(15) Phenomena of stress concentration

Stress concentration occur in members where we have discontinuity.
If we have cutouts, then ~~we~~ the flow of stresses change, and we get stress concentration.
We discussed St. Venant's principle.



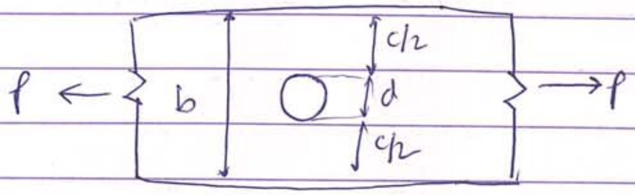
At point of application, the area squishes whereas towards the center, the lines are vertical and more closer to each other.
Towards end we have large stress concentration.
- Concentrated loads result in large stresses in the vicinity of the load application point
- Stress and strain distribution became uniform at a relatively short distance from the load application points (least lateral dimension).

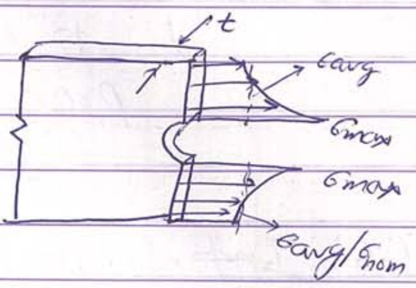


h is the least lateral dimension (h < t)
So, uniformity prevails at the distance of least lateral dimension.

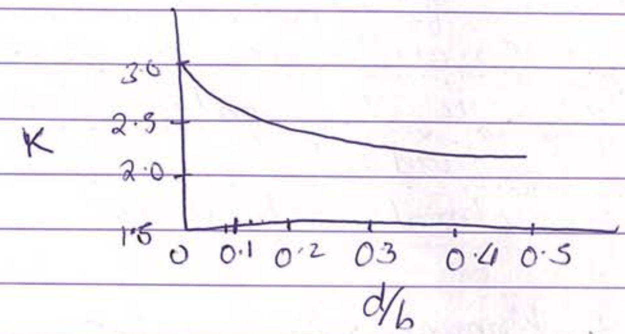
Example

- Presence of notches/holes





- Discontinuities of cross section may result in high localized or concentrated stress



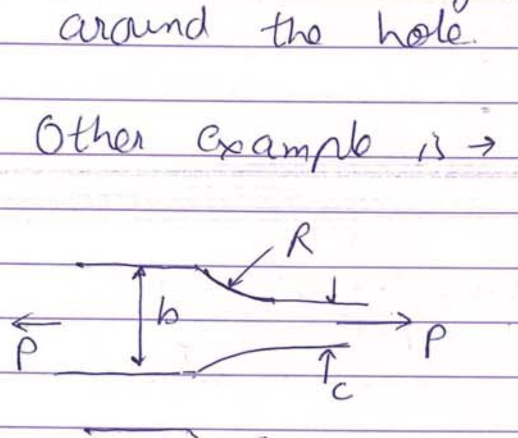
$$K = \frac{\sigma_{max}}{\sigma_{nom}}$$

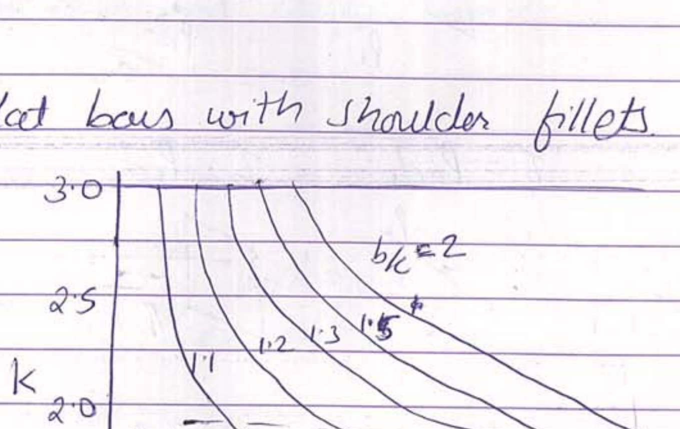
$\sigma_{nom} = P/bt$

t = thickness.

Near edge of hole we have σ_{max}, and towards end stresses reduce.
So, when b is very large, we have high stress concentration around the hole.

Other example is → flat bars with shoulder fillets





region of failure

$k = \frac{\sigma_{max}}{\sigma_{nom}}$ $\sigma_{nom} = P/ct$ t = thickness

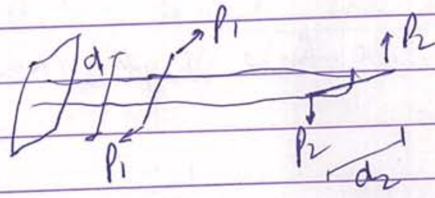
So, we have gradual reduction.
If we have sharp transition, stress concentration are going to be even higher.

Generally these charts are present that can be used to calculate thickness, and material to be provided.

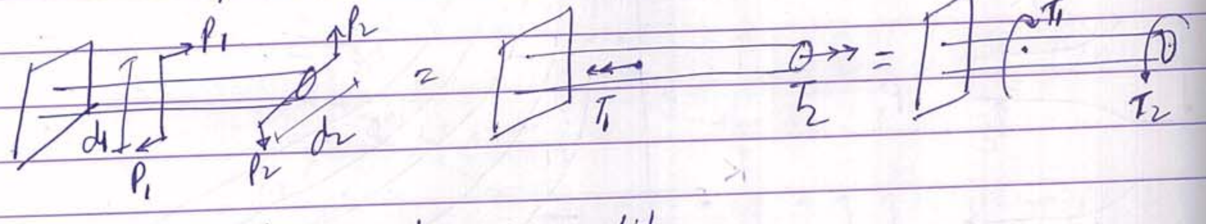
(16) ~~Intor~~ Introduction to Torsion

We looked at member ~~a~~ having axial tension. Here we will twist it. So, we will see effect of torque on the structure. There are difference b/w torque and axial tension but there are analogies that will help to relate the structural behaviour under ~~torq~~ torsion to structural behaviour under axial loading.

Ex of Torsion: screw driver, Wringing water from cloth, opening tap / bottle, etc.



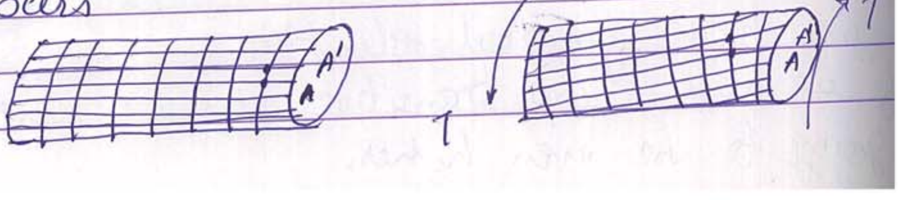
$T_1 = P_1 d_1$ $\quad$ $T_2 = P_2 d_2$



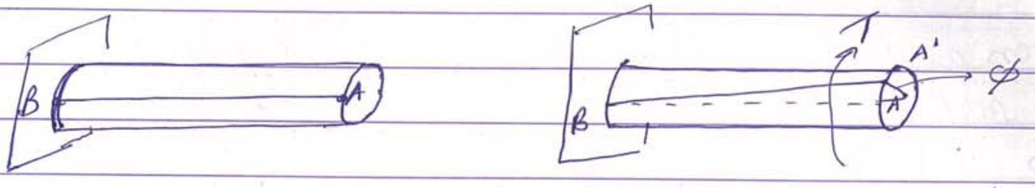
- Torque is vector quantity
- Use right hand rule to get dir^n of torque.

How do structure behave under torsion?

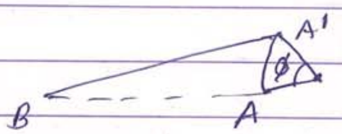
· Circular bars



Due to torque applied in circular shaft AA' has shifted along the same ~~to~~ circumferential line. So,



ϕ = angle of Twist.



In case of non-circular members we get <u>warping</u>.

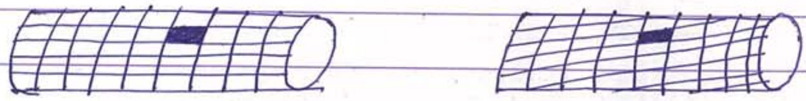
In case of circular bars we don't have warping. So, it is called <u>Pure Torsion</u>. So, points on circular cross section remain on same cross section.

<u>Pure Torsion</u>

The center point does not move anywhere. But as we go towards circumference the displacement increases.

Pure shear condition

In pure torsion elements only undergo a change in shape, and minor contraction or elongation.

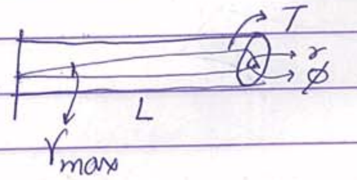


As we undergo only change in shape, these elements are in <u>pure shear</u>.

Imp All elements are in a state of <u>Pure Shear</u>.

(17) Uniform Torsion: Part 1 (Shear Mechanism + Torsion Formula)

Shear stress and shear strain are going to be higher on outer surface and slowly going to 0 as it approach towards center.



ϕ = angle of Twist

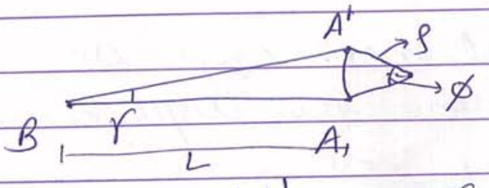
γ_{max} = max amount of shear

At exterior surface we have γ_{max} and at inner we will have γ.

→ Inner section

inner radius = ρ

ϕ remains same for inner and outer section.



$$\tan\gamma = \frac{AA'}{L} \Rightarrow \gamma = \frac{AA'}{L} \quad \text{(small angle)} \quad -①$$

$$AA' = \rho\phi \quad -②$$

So,

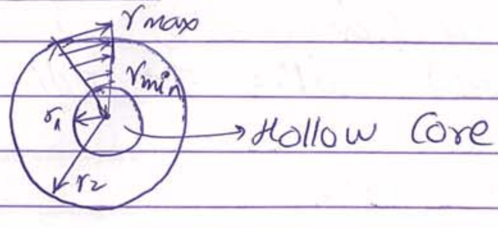
$$\boxed{\gamma = \frac{\rho\phi}{L} \Rightarrow \gamma_{max} = \frac{r\phi}{L}}$$

$\rho = 0$, at center so $\gamma = 0$ at center.

$$\boxed{\gamma = \frac{\rho}{r}\gamma_{max}}$$

This was for solid circular shaft.

Circular tubes



$$\gamma_{min} = \frac{r_1 \phi}{L}$$

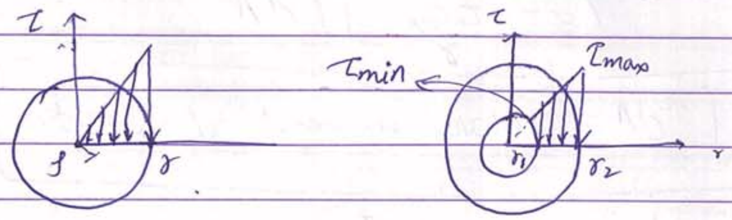
$$\gamma_{max} = \frac{r_2 \phi}{L}$$

$$\boxed{\gamma_{min} = \frac{r_1}{r_2} \gamma_{max}}$$

There must be associated change in stresses.

Shear Stresses

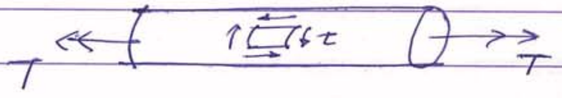
$\tau = G\gamma$



for solid $\tau_{max} = G \times \gamma_{max} = \frac{G \times r \phi}{L}$; $\boxed{\tau = \frac{G \rho \phi}{L}}$

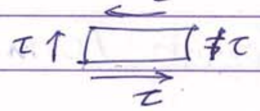
for hollow $\tau_{max} = \frac{G r_2 \phi}{L}$; $\tau_{min} = \frac{G r_1 \phi}{L}$

$$\boxed{\tau_{min} = \frac{r_1}{r_2} \tau_{max}}$$



Here all elements are in pure shear.

So,



At exterior, all are τ_{max}.
At interior, it reduces linearly.

Torsion formula

It tries to establish relation b/w T & ϕ → angle of twist

Force acting on element $= \tau dA$

Distance from center $= \rho$.

Moment $dM = \cancel{\Sigma\rho}\ (\tau dA) \times \rho$

The net moment is torque.

So,

$$T = \int_A dM = \int \tau \rho \, dA.$$

We know $\dfrac{\tau}{\tau_{max}} = \dfrac{\rho}{r}$

So, $T = \int \dfrac{\tau_{max}}{r} \rho^2 dA$

$$T = \frac{\tau_{max}}{r} \int \rho^2 dA$$

$\boxed{\int \rho^2 dA = \text{Polar moment of inertia}} = J$

$$\boxed{T = \frac{\tau_{max} J}{r}}$$

$\Rightarrow \boxed{\tau_{max} = \dfrac{Tr}{J}}$

So, $\tau = \dfrac{T\rho}{J}$ (for internal radius)

We know $\boxed{\tau_{max} = \dfrac{G r \phi}{L}}$

So $\dfrac{G r \phi}{L} = \dfrac{Tr}{J}$

$\Rightarrow \boxed{\phi = \dfrac{TL}{GJ}}$ Torsion formula.

It relates ϕ to T.

NB! $\rightarrow$ J (solid shafts) $= \dfrac{\pi r^4}{2}$; J (hollow shaft) $= \dfrac{\pi r_2^4 - \pi r_1^4}{2}$

Summary →

$\gamma = \frac{\rho\phi}{L}$

$\gamma_{max} = \frac{\rho r}{L}$

~~r_{min}~~ hollow ⇒

$\gamma_{min} = \frac{r_1 \phi}{L} = \frac{r_1}{r_2}\gamma_{max}$

$\tau_{min} = \frac{r_1}{r_2}\tau_{max}$

$\tau_{max} = \frac{Tr}{J}$

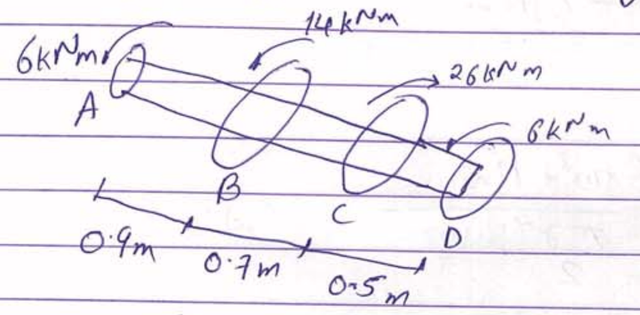
$\tau = \frac{T\rho}{J} = \frac{\rho}{r}\tau_{max}$

$\tau = G\gamma$

$\phi = \frac{TL}{GJ}$

(18) Uniform Torsion: Part 2 (Problem Example)

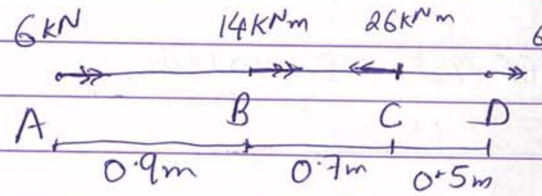
Purpose of disc is to apply torque easily



- Shaft BC is hollow, d_{inner} = 90mm, d_{outer} = 120mm
- Shaft AB & CD are solid, diameter d.
  - find max shearing stress in BC
  - Required diameter d of AB & CD if allowable shearing stress in them is 65 MPa

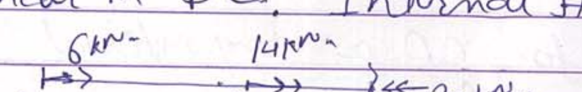
$T_A = 6$ kNm $\quad T_B = 14$ kNm

$T_C = 26$ kNm $\quad T_D = 6$ kNm



Bar is in equilibrium.

→ Max shear in BC. Internal ~~stress~~ torque in BC



$T = 20$ kNm. $\quad \tau_{max} = \frac{Tr}{J} = \frac{T \times r_2}{J}$

$$J = \frac{\pi(r_2^4 - r_1^4)}{2} = \frac{\pi}{2} \times \left(\frac{(120)^4 - (90)^4}{2^4}\right)$$

$$= \frac{222660379.323}{2^4} = \frac{2.22 \times 10^8}{2^4} \ \underline{mm^4}.$$

$$\tau_{max} = \frac{Tr}{J} = \frac{20 \times 10^3\, Nm \times 0.12\, m \times 2^4}{2.22 \times 10^8 \times 10^{-12}\, m^4 \times 2}$$

$$= \frac{20 \times 10^3 \times 0.12 \times 10^4}{2.22} \ \cancel{N/m^2} \times 2^3\, N/m^2$$

$$= 1.08108 \times 10^7 \ \cancel{N/m^2} \times 2^3\, N/m^2$$

$$= \cancel{10.81\ MPa}.$$

$$= \cancel{172.97\ MPa}\ \ 86.4\ MPa$$

(2) $\tau_{max} = 65\, MPa.$

For AB $\quad A \xrightarrow{6kNm} \} \leftarrow 6\,kNm$

$$T_{AB} = 6\,kNm.$$

$$\tau = \frac{Tr}{J} = \frac{6 \times 10^3\, Nm \times r}{\frac{\pi r^4}{2}}$$

$$65 \times 10^6\, N/m^2 = \frac{6 \times 10^3}{\frac{\pi r^3}{2}}\, N/m^2$$

$$r^3 = \frac{6 \times 10^3 \times 2}{65 \times 10^6}\, m^3$$

$$= 12000 / 65 \times 10^6 = 0.000184615$$

$$r^3 = 184615.3846\, mm^3$$

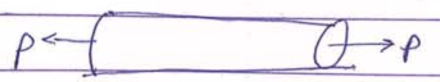
$$r = 56.94$$

$$d_{AB} = 113.88\, mm$$

Similar will be for CD as ~~torq~~ internal torque for CD is same as AB.

(19) Nonuniform Torsion: Part 1 (Analogies with axial loading + Indeterminate structures)

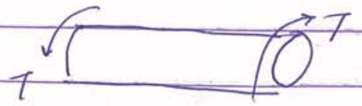
- Axial



$$\delta = \frac{PL}{AE}$$

AE → Axial Rigidity

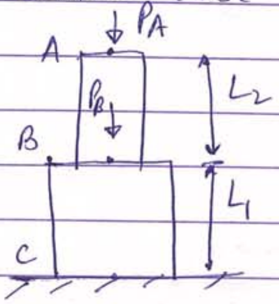
- Torsional (Pure shear)



$$\phi = \frac{TL}{GJ}$$

GJ → Torsional Rigidity

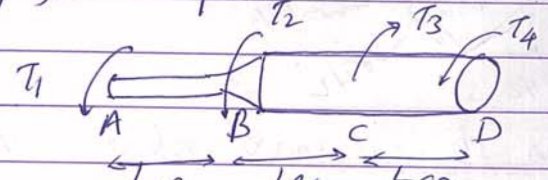
For axial case we had



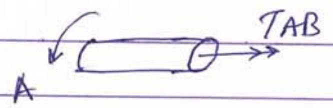
$$\delta_{AC} = \sum \delta_i = \sum_{i=1}^{n} \frac{N_i L_i}{A_i E_i}$$

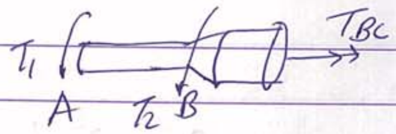
N → Internal forces

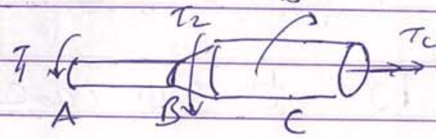
Similarly for torques



We can cut sections







$T_{int,1} = T_{AB}$

$T_{int,2} = T_{BC}$

$T_{int,3} = T_{CD}$

ϕ_1 = angle of twist b/w A & B $= \dfrac{T_{int,1} \times L_{AB}}{J_{AB} \times G_{AB}}$

ϕ_2 = angle of twist b/w B & C $= \dfrac{T_{int,2} \times L_{BC}}{J_{BC} \times G_{BC}}$

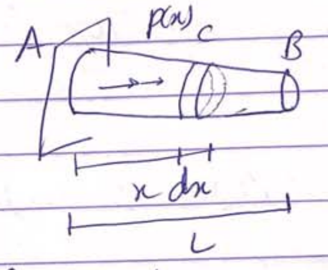
ϕ_3 = angle of twist b/w C & D $= \dfrac{T_{int,3} \times L_{CD}}{J_{CD} \times G_{CD}}$

$$\phi_{total} = \phi_1 + \phi_2 + \phi_3$$

$$= \sum \phi_i$$

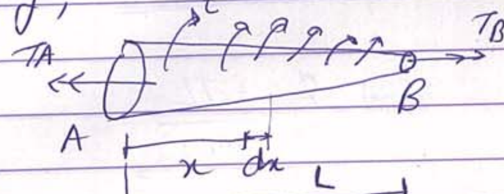
$$= \sum \frac{T_{int,i} \times L_i}{J_i \times G_i}$$

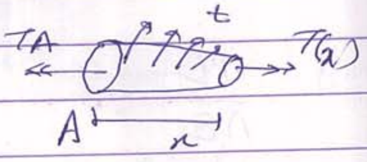
For axial case



$$\delta = \int_0^L d\delta = \int_0^L \frac{N(x)\,dx}{E A(x)}$$

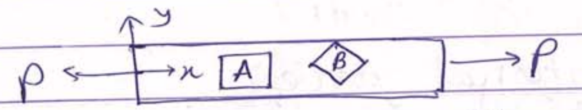
Similarly,

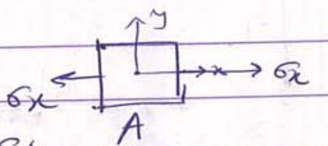


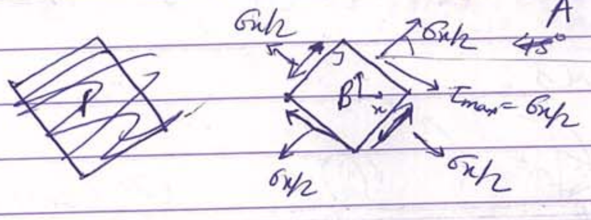


$$\boxed{\phi = \int_0^L \frac{T_{int}(x)\,dx}{J(x)\,G}}$$

For axial case





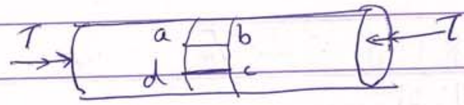


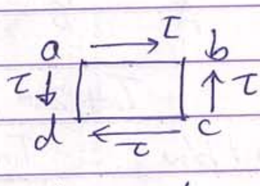
$\theta = 45°$

At $\theta = 45°$, shear stress is max value $= \sigma x/2$.

magnitude of τ is $\sigma x/2$.

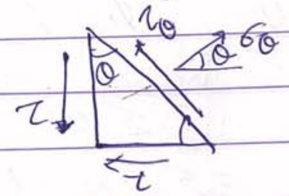
For torsion case





in 2D

Now, if we see at an ~~ant~~ angle



~~$\sigma_\theta = 2\tau$~~ $\quad \sigma_\theta \cos\theta - \tau_\theta \sin\theta - \tau = 0 \quad -(1)$

$\sigma_\theta \sin\theta + \tau_\theta \cos\theta - \tau = 0 \quad -(2)$

$\sigma_\theta \cos\theta - \tau_\theta \sin\theta = \sigma_\theta \sin\theta + \tau_\theta \cos\theta$

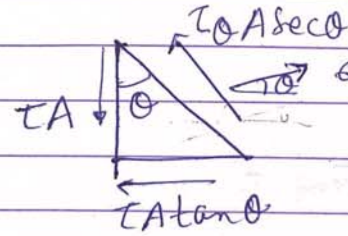
$\sigma_\theta(\cos\theta - \sin\theta) = \tau_\theta(\sin\theta + \cos\theta)$

$$\sigma_\theta (\cos^2\theta - \sin^2\theta) = \tau_\theta (1 + 2\sin\theta\cos\theta)$$
$$\sigma_\theta (2\cos^2\theta - 1) = \tau_\theta (1+\sin 2\theta)$$
$$\sigma_\theta \cos 2\theta = \tau_\theta (1+\sin 2\theta)$$
$$\sigma_\theta = \tau_\theta \left(\frac{1+\sin 2\theta}{\cos 2\theta}\right)$$

$$\cancel{\sigma_\theta \sin\theta}\ \tau_\theta \left(\frac{1+\sin 2\theta}{\cos 2\theta}\right) \times \sin\theta + \tau_\theta \cos\theta = \tau$$

$$\tau_\theta \left((1+\sin 2\theta) \times \sin\theta + \cos\theta \cos 2\theta \right) = \tau_\theta \cos 2\theta$$

$$\tau_\theta (\sin\theta + \sin\theta \sin 2\theta + \cos\theta \cos 2\theta) = \tau_\theta \cos 2\theta$$
$$\tau_\theta (\sin\theta + \cos\theta) = \tau_\theta \cos 2\theta$$
$$\tau_\theta = \frac{\tau \cos 2\theta}{(\sin\theta + \cos\theta)}$$
$$\tau_\theta = \tau (\cos\theta - \sin\theta)$$



$\Sigma f_x = 0 \qquad \Sigma f_y = 0$

$$\sigma_\theta A\sec\theta\cos\theta - \tau_\theta A\sec\theta\sin\theta - \tau A\tan\theta = 0 \quad -(1)$$
$$\sigma_\theta A\sec\theta\sin\theta + \tau_\theta A\sec\theta\cos\theta - \tau A = 0 \quad -(2)$$

$$\sigma_\theta A - \tau_\theta A\tan\theta = \tau A\tan\theta$$
$$\sigma_\theta A\tan\theta + \tau_\theta A = \tau A$$

$$\sigma_\theta A - \tau_\theta A\tan\theta = \sigma_\theta A\tan^2\theta + \tau_\theta A\tan\theta$$
$$\sigma_\theta A(1-\tan^2\theta) = 2\tau_\theta A\tan\theta$$
$$\sigma_\theta = \frac{2\tau_\theta\tan\theta}{1-\tan^2\theta}$$

$$\frac{2\tau_\theta\tan^2\theta}{1-\tan^2\theta} + \tau_\theta = \tau$$

$$\frac{\tau_\theta(2\tan^2\theta + 1 - \tan^2\theta)}{1-\tan^2\theta} = \tau$$

$$\tau_\theta\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \tau$$

$$\tau_\theta\left(\frac{1}{\cos 2\theta}\right) = \tau \Rightarrow \boxed{\tau_\theta = \tau\cos 2\theta}$$

$$\sigma_\theta - \tau_\theta \tan\theta = \tau \tan\theta$$

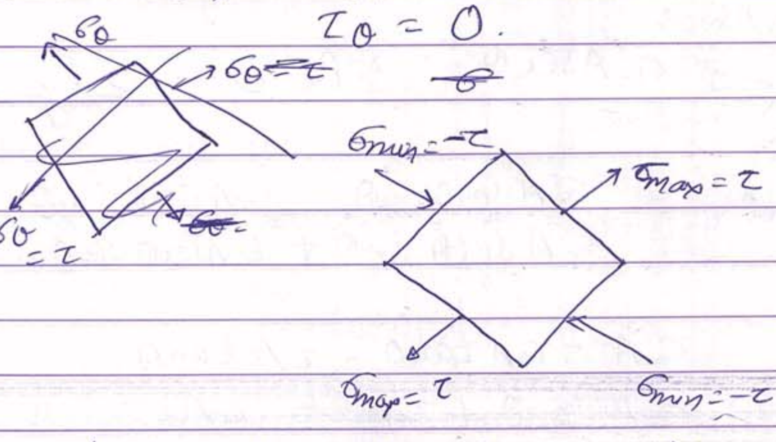
$$\tau_\theta = \tau \cos 2\theta$$

$$\sigma_\theta = \tau \cos 2\theta \tan\theta + \tau \tan\theta$$
$$= \tau \tan\theta (2\cos^2\theta) = 2\sin\theta\cos\theta\,\tau$$

$$\boxed{\sigma_\theta = \tau \sin 2\theta}$$

So, for an angle $\boxed{\sigma_\theta = \tau\sin 2\theta,\ \tau_\theta = \tau\cos 2\theta}$

In longitudinal portion, we had only shear stress
when $\theta = 0$ $\sigma_\theta = 0$, $\tau_\theta = \tau$
Now, when $\theta = 45^\circ$, $\sigma_\theta = \sigma_{max} = |\tau|$
$\tau_\theta = 0$.



So, there is failure if material is weak in ~~normal~~ tension
eg: Brittle materials (like classroom chalk)

Brittle materials are weak in tension. Hence under pure torsion, it tends to fail along the 45° plane.

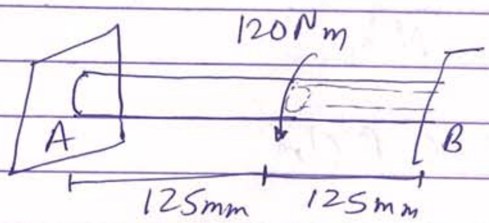
Statically Indeterminate Case

$$-T_A + T_B = T_0$$

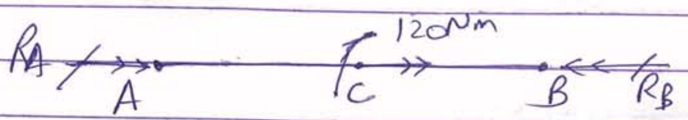
$\phi_{AB} = 0 \Rightarrow$ Compatibility eq^n.

$(\phi_{AC} + \phi_{CB} = 0)$

(20) Non-uniform Torsion: Application Example



- Solid core has dia 25mm
- Hollow core has dia 15mm
- Find support reactions

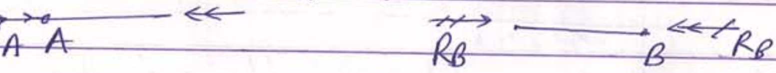


~~$R_A + 120$~~ $R_A - R_B = 120$

$\phi_{AB} = 0$.

$\phi_{AC} + \phi_{CB} = 0$

For ϕ_{AC} $\quad T_{int} = R_A$



$$\phi_{AC} = \frac{T_{int,AC} \times L_{AC}}{J_{AC} \times G} \qquad \phi_{CB} = \frac{T_{int,BC} \times L_{BC}}{J_{BC} \times G}$$

(For compression, we take negative, similar will be case here)

$$\phi_{AC} = -\frac{R_A \times L_{AC}}{J_{AC} \times G} \qquad \phi_{CB} = -\frac{R_B \times L_{BC}}{J_{BC} \times G}$$

$\phi_{AB} = 0$

$$\frac{R_A \times L_{AC}}{J_{AC} \times G} + \frac{R_B \times L_{BC}}{J_{BC} \times G} = 0$$

(Same material, so same G), & $L_{AC} = L_{BC}$

$$\frac{R_A}{J_{AC}} = -\frac{R_B}{J_{BC}}$$

$$J_{AC} = \frac{\pi}{2}\left(\frac{25}{2}\right)^4 \qquad J_{BC} = \frac{\pi}{2}\left(\left(\frac{25}{2}\right)^4 - \left(\frac{15}{2}\right)^4\right)$$

$J_{AC} \Rightarrow$ ~~306796.1579~~ $\quad J_{BC} = 33379.4219$

$J_{AC} = 38349.51$

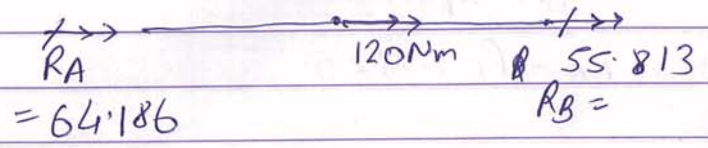
$$R_A = -\frac{38349.51}{33379.4219} R_B \qquad R_A = -1.15 R_B$$

$$-1.15R_B - R_B = 120$$

$$R_B = -\frac{120}{2.15} = -55.813 \text{ Nm}$$

$$R_A = -1.15R_B = 64.186 \text{ Nm}$$

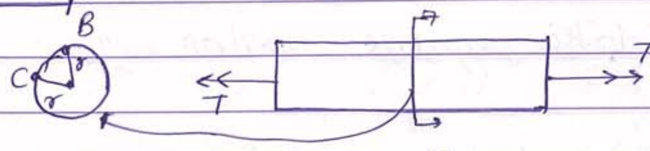
So, our dir^n of R_A is correct & R_B is wrong



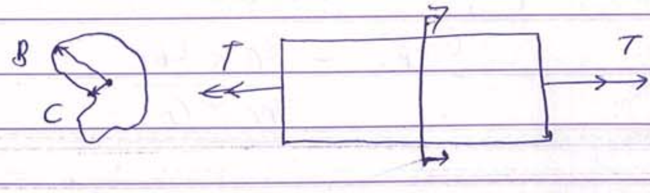
(21) Torsion in Thin walled Tube + Stress Concentration

We will see shear flow & stress concentration.

Recap



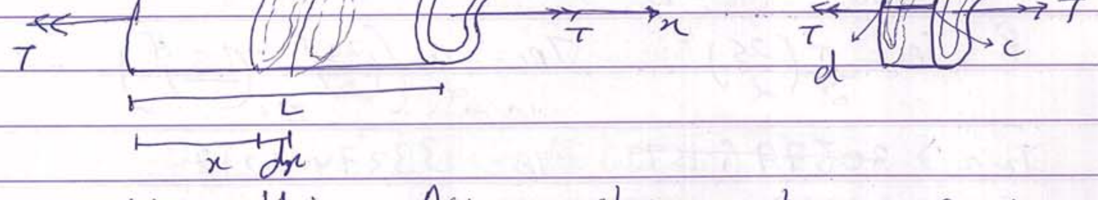
$\tau_B = \tau_C$ (as $r_B = r_C = r$)



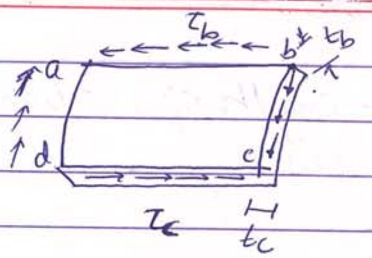
$\tau_B \neq \tau_C$ ($r_B \neq r_C$)

Prismatic bars but different crossections.
As r_B & r_C are not equal so shear stress is not equal

Thin walled Tubes

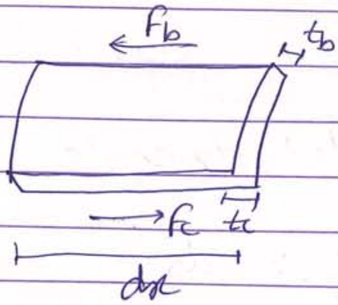


Wall is thin - Assume shear stress is constant through the thickness



thickness varies

From a to b we have τ_b and from c to d we have τ_c. From b to c we have transition from τ_b to τ_c.
As wall is very very thin so internal and outer stresses are the same.



$F_b = \tau_b \times t_b \times dx$
$F_c = \tau_c \times t_c \times dx$

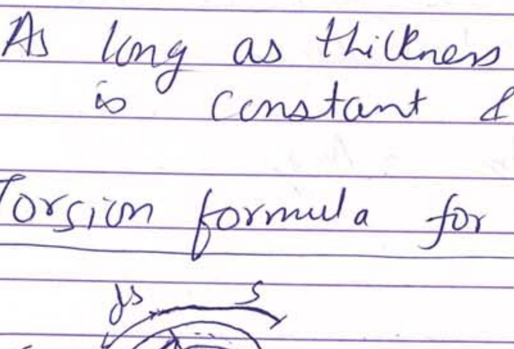
There should be eq^m. So $\Sigma F_x = 0$.

$\Rightarrow \tau_b \times t_b \times dx = \tau_c \times t_c \times dx$
$\Rightarrow \boxed{\tau_b t_b = \tau_c t_c} = q$ (Shear flow)

So, q remains constant across the entire cross section. It is called as shear flow.

As long as thickness is small, product of τ & t is constant & called shear flow.

Torsion formula for thin walled tube



ds → small element along median line
$dT = \tau \times t \times ds \times r$
$(\because \tau t = q)$
$dT = q\, r\, ds$.

$dT = q r\,ds$ (l_m = length of median line)

$\Rightarrow T = \int_0^{l_m} q r\,ds$

q is constant $(\tau \times t)$

$\Rightarrow T = q \int_0^{l_m} r\,ds$

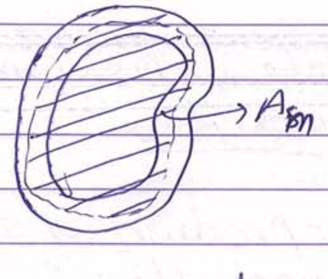
It is difficult to calculate $\int_0^{l_m} r\,ds$ r changes at every point.

For our area of shaded triangle

$A_s = \frac{1}{2} \times r \times ds$

So, $r\,ds = 2A_s$

So, $\int_0^{l_m} r\,ds =$ Total area of section $\times 2$
(enclosed by median line)
$= 2 \times A_m$



Am is not ~~the~~ cross sectional area but area enclosed by median line.

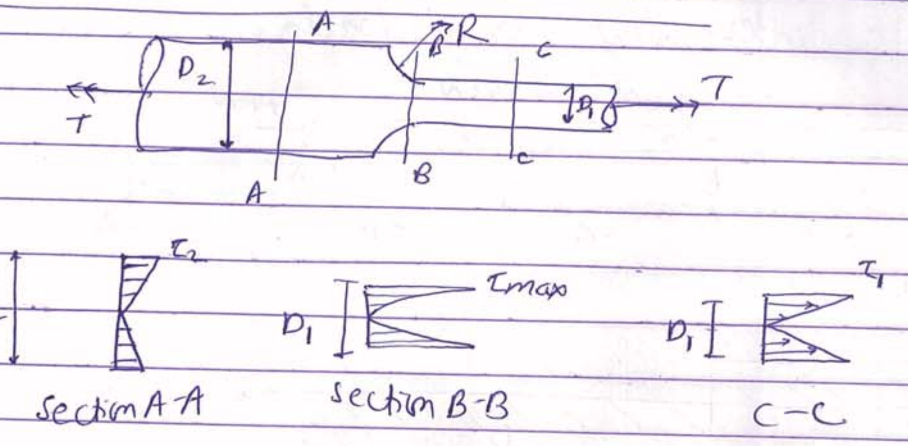
So, $T = q \int_0^{l_m} r\,ds = q \times 2A_m = 2A_m q$

So, $\boxed{q = \frac{T}{2A_m}}$

$\Rightarrow \tau \times t = \frac{T}{2A_m} \Rightarrow \boxed{\tau = \frac{T}{2A_m \times t}}$

* assuming t is same through out

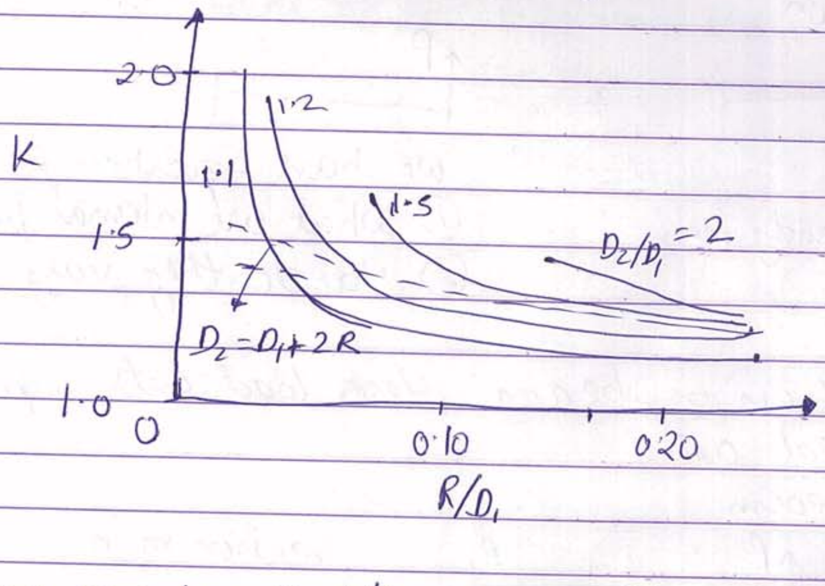
* Stress concentration in Torsion



~~we~~ we ~~k~~ expect it to fail near section B-B. At end ~~of~~ point in B-B, our stresses ~~show~~ shoot up due to discontinuities, so, we have τ_{max}

$$\tau_{max} = K \times \tau_{nominal} \qquad \tau_{nominal} = \frac{16T}{\pi D_1^3}$$

k = stress amplification factor (always higher than 1) & it depends upon fillet radius

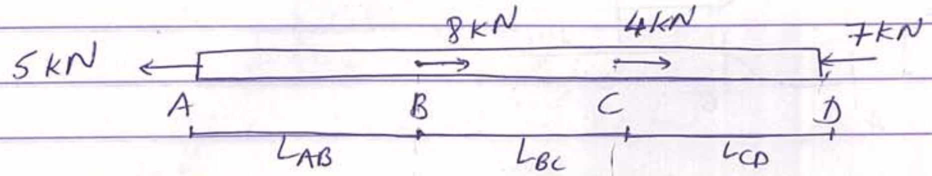


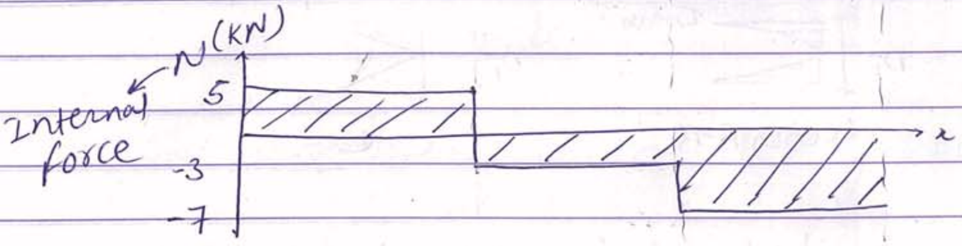
R = fillet radius
D_1 = smaller dia
D_2 = bigger dia

(Hollow circular shafts are not thin walled)
For torsion for non-circular ~~shafts~~ sections, we study in advanced solid mechanics course.
Warping happens in non-circular sections

(22) Intro to SFD, BMD & Sign Conventions

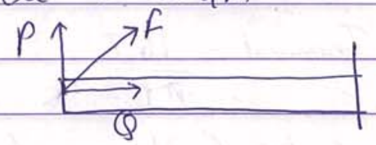
- Bars with intermediate axial loads



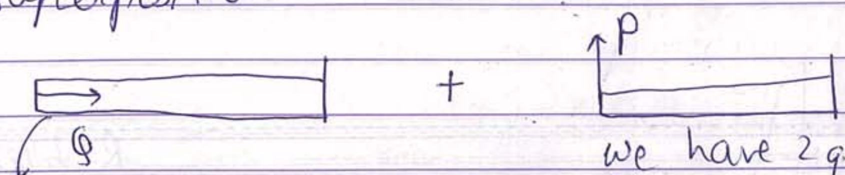


Axial force Diagram

Now consider a beam



F can be splitted to P & Q & using principle of superposition.

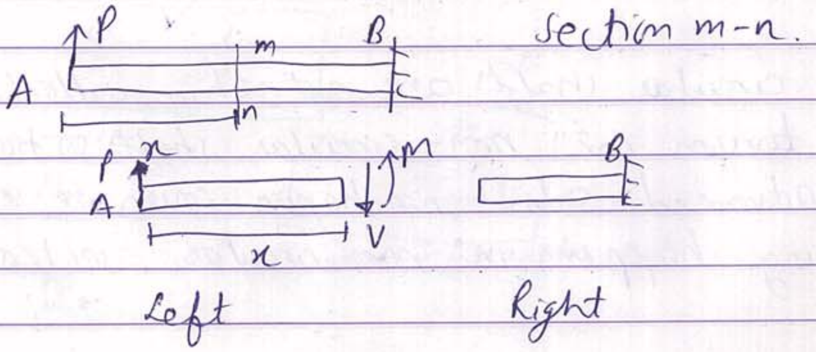


we have already seen

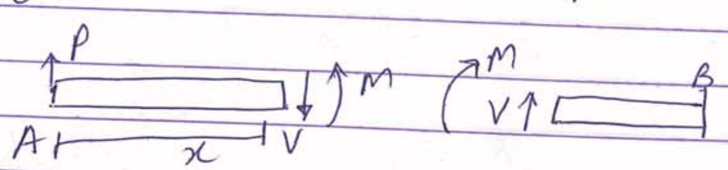
we have 2 questions for this
① What are internal forces?
② How do they vary?

So, we will focus on beams where load acts perpendicular to longitudinal axis.

Consider the beam



At section we have shearforce V & bending moment M. On right side we must have equal & opposite force & moment.



For eq^m,

$$\Sigma F_y = 0 \Rightarrow V = P. \quad V = \text{shear force}$$
$$M = \text{bending moment.}$$

$$\Sigma M = 0 \Rightarrow Px + V \cdot 0 - M = 0$$
$$M = Px$$

So, shear does not vary & moment is a fn of x.
The value of V & M depend upon nature of loading & type of structure that we have.

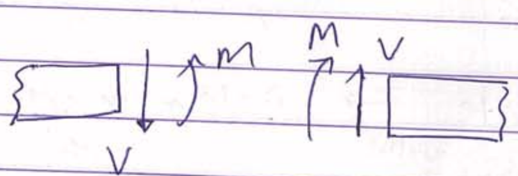
Variation of ~~beam~~ shear force (V) & Bending Moment (M) along the structure are represented using SFD & BMD.

SFD & BMD helps to find maximum shear and moment values for designing members.

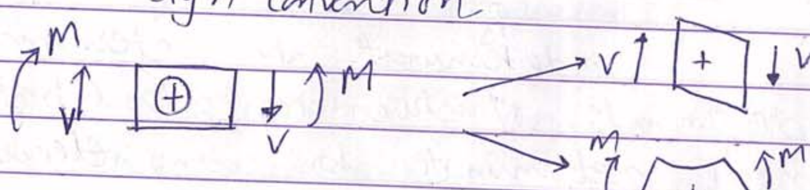
<u>Sign convention</u> ~~→ Sagging~~ ◡ → Sagging (Smiley)
◠ → Hogging

+ve shear - Tends to rotate element clockwise

+ve moment - Tends to cause sagging (converts the beam to a trough)

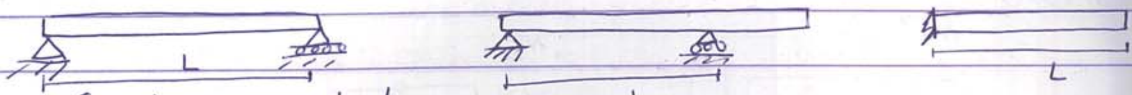


+ve sign convention



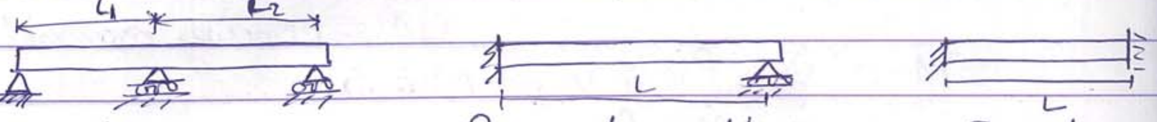
(23) Types of Beams

Determinate



Simply supported   Overhang   Cantilever

Indeterminate



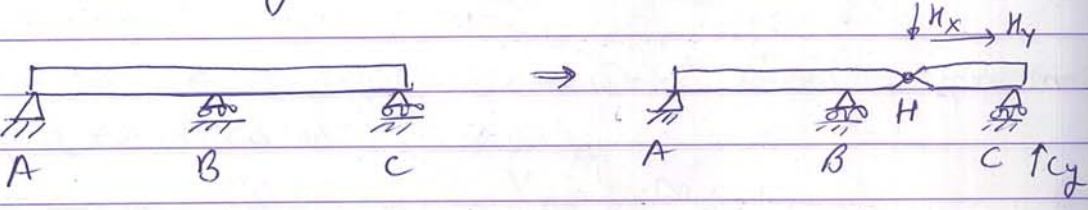
Continuous   Propped cantilever   Fixed

For statically determinate we have 2 eq^n's & 2 unknowns.

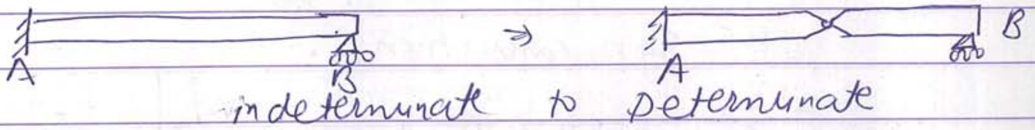
For continuous beam we have 3 ~~eq^{n}s~~ unknowns & 2 eq^n's.
For propped cantilever ~~&~~ we have 3 unknowns & ~~3~~ 2 eq^n
For fixed we have ~~3~~ 4 unknowns & 2 eq^n's of eq^n.

But we can solve indeterminate beams.

We can make indeterminate structures as determinate by introducing an internal hinge.
(Hinge can't restrict moment ~~s~~, so it does not have bending moment at that location)



Indeterminate → Determinate



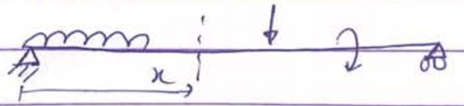
indeterminate to determinate

→ Indeterminate structures can often (but not always) be made determinate by using releases.

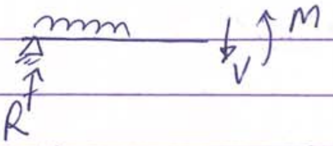
(24) Method of Sections (Basic Fundamentals)

Recap

Consider the beam

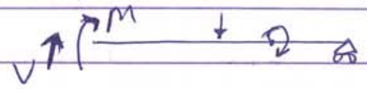


Left Part



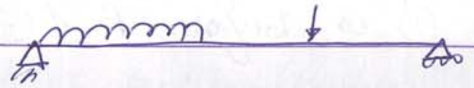
We have marked the convention for V & M.
V or M may or may not come out as +ve.

Right Part

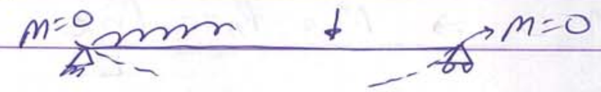


Whethere we cut from right or left V & M should come out one and the same.

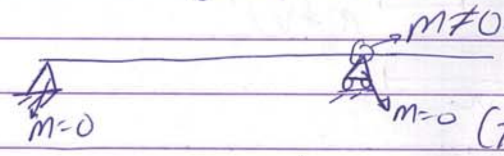
Consider a simply supported beam



When pin ends / roller ends are at the very end of the beam, then it will not develop any bending moment.
As long as it is a pin/roller support, it allows rotation. As we have rotation, the moment does not develop.



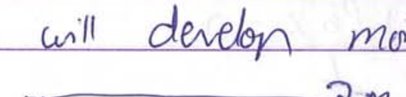
This is not true if pin/roller is at some mid point.



(The pin can't take a moment)

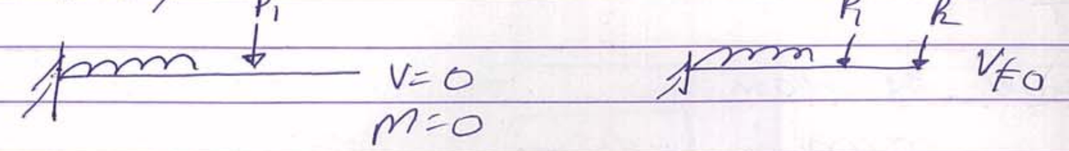
But in beam, we will have a finite value of m

If we have a external moment at pin (which is at end) then beam will develop moment.



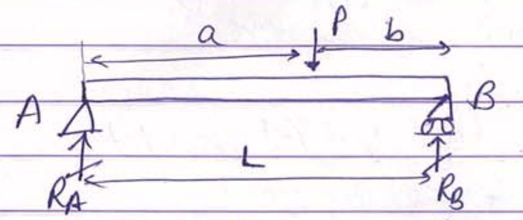
(For pin here also bending moment is 0)

Free end of cantilever will have $V=0$ & $M=0$ (if there is no external load/moment at free end)



(25) Method of sections (Problem 1)

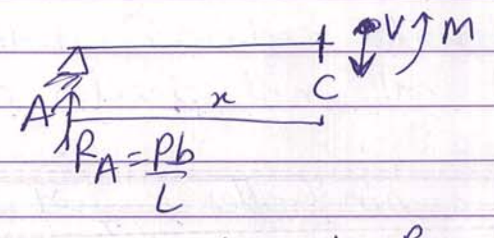
Draw SFD & BMD



$R_A = \frac{Pb}{L}$; $R_B = \frac{Pa}{L}$

We take sections at point of discontinuities. Here it is at P, so ① is before P & ② is after P.

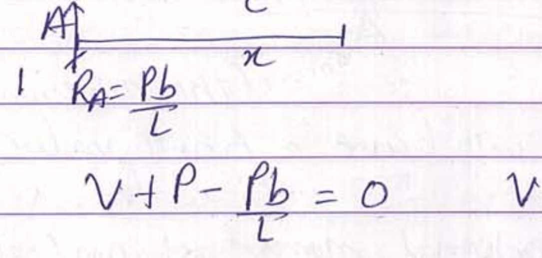
Section ①



$\underline{V}$ $R_A - V = 0 \Rightarrow V = R_A = \frac{Pb}{L}$

$\underline{M}$ $-R_A x + M = 0 \Rightarrow M = R_A x = \frac{Pbx}{L}$

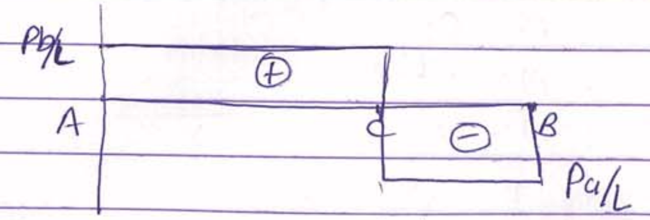
Section ②

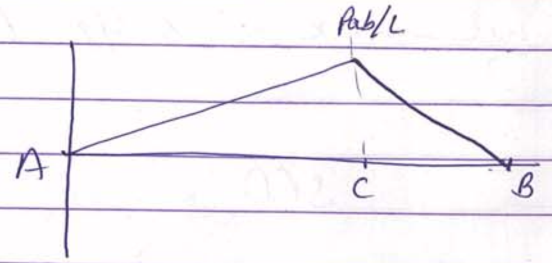


$\underline{V}$ $V + P - \frac{Pb}{L} = 0$ $\quad V = -\frac{Pa}{L}$

$\underline{M}$ $M + P(x-a) - \frac{Pb}{L}x = 0$

$$M = Pa - Px\left(1 - \frac{b}{L}\right) = Pa - \frac{Pax}{L}$$

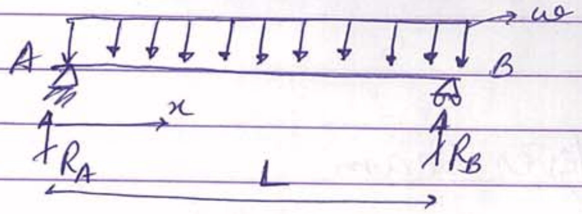




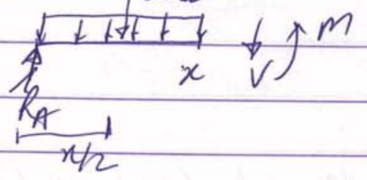
The bending moment is maximum at point C. And at that point shear goes from +ve to -ve.

We can also solve by taking sections from right instead ~~to~~ of from left

(26) Method of sections (Problem 2)



$R_A = R_B = \frac{wL}{2}$; As no



$$R_A - \frac{wx}{\cancel{2}} - V = 0$$

$$V = \cancel{\frac{w}{2}(l-x)}\ \frac{wl}{2} - wx$$

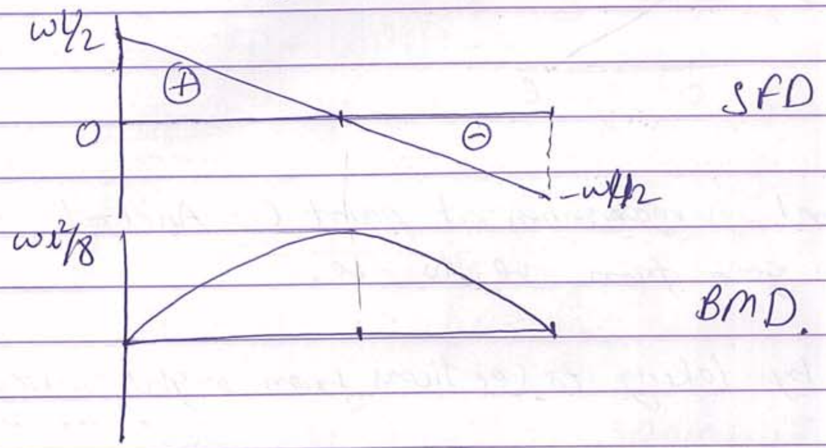
$$M + \frac{wx}{\cancel{2}} \times \frac{x}{2} - \frac{wL}{2} \times x = 0$$

$$M = \cancel{\frac{w}{2}(x^2 - lx)}\ \frac{wlx}{2} - \frac{wx^2}{2}$$

$V = \frac{wL}{2} - wx \qquad M = \frac{wLx}{2} - \frac{wx^2}{2}$

@ $x = 0$, $V = wL/2$, $M = 0$
@ $x = L/2$, $V = 0$, $M = wL^2/8$
@ $x = L$, $V = -wL/2$, $M = 0$

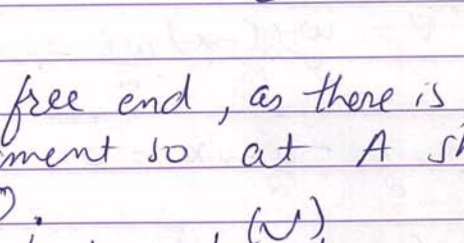
For max $\frac{dM}{dx} = 0 \Rightarrow \frac{wL}{2} - \frac{2wx}{2} \Rightarrow x = L/2$, $M_{max} = wL^2/8$



NB! → max ~~shear~~ Bending moment occurs when shear force is 0.

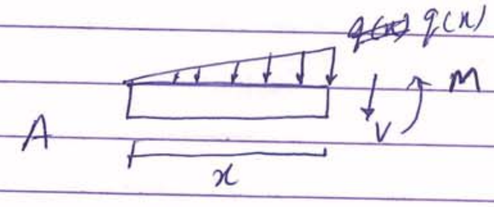
(27) Problem 3

We have a cantilever beam



A is free end, as there is no external force or bending moment so at A shear force & bending moment is 0.
Instead of trough (∪) we will get a valley (∩) here due to type of loading

Much easier to solve when going from left to right.

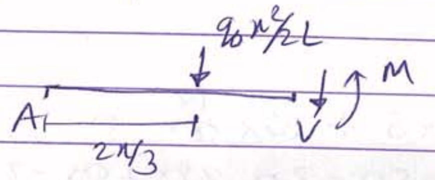


$$V + \frac{1}{2} \times q(x)\, x = 0$$

$$V = -\frac{q(x)\, x}{2}$$

$q(x) = \frac{q_0 x}{L}$ $\qquad V = -\frac{q_0 x^2}{2L}$

The load $\frac{q_0 x^2}{2L}$ will act at a distance of $\frac{2}{3}x$ from point A.



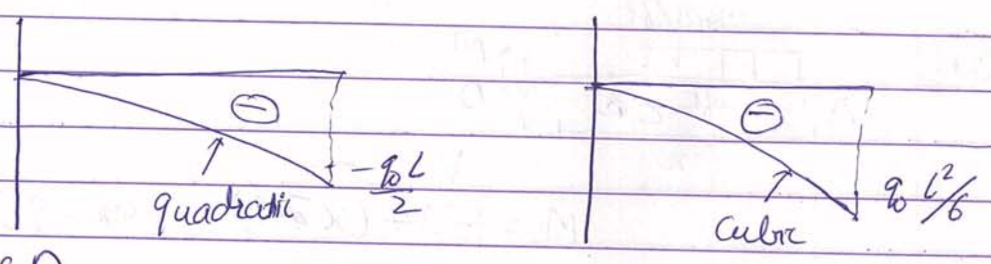
$$M + \frac{q_0 x^2}{2L} \times \frac{2x}{3} = 0$$

$$M = \frac{-q_0 x^3}{6L}$$

$V = \frac{-q_0 x^2}{2L}$, $M = \frac{-q_0 x^3}{6L}$

@ $x = L$, $V = -\frac{q_0 L}{2}$, $M = -\frac{q_0 L^2}{6}$

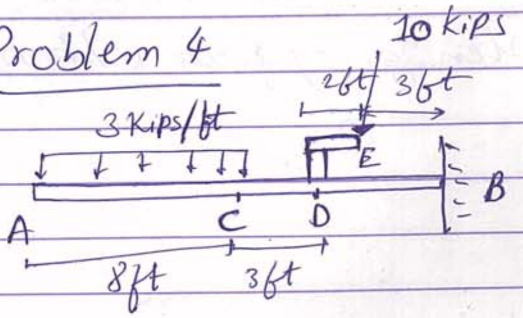
@ $x = 0$, $V = 0$, $M = 0$

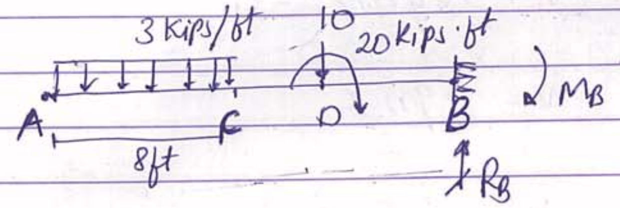


SFD

For concave up or down $\Rightarrow \frac{dM}{dx} = -\frac{q_0 x^2}{2L}$, our slope becomes more -ve as we move towards L.

(28) Problem 4





We have to first calculate R_B & M_B

So, $-8 \times 3 - 10 + R_B = 0$

$R_B = 34$ kips

For moment,

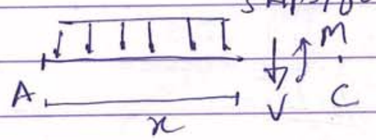
$-M_B - 20 + 10 \times 5 + 24 \times \cancel{10}\ 12 = 0$

$M_B = \cancel{240 + 50 - 20}\ 288 + 50 - 20$

$= \cancel{270}\ 318$ kips·ft.

Now, we will draw SFD & BMD. We will have sections at point of discontinuity.

For section ①

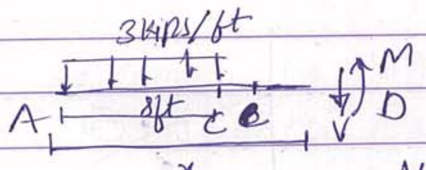


$3 \times x + V = 0$

$V = -3x$

$M = -3x^2/2$

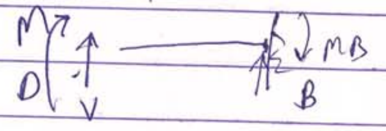
Section ②



$V = -24$

$M = -24(x - 4) = -24x + 96$

Section③ . ~~we draw~~ from right to left

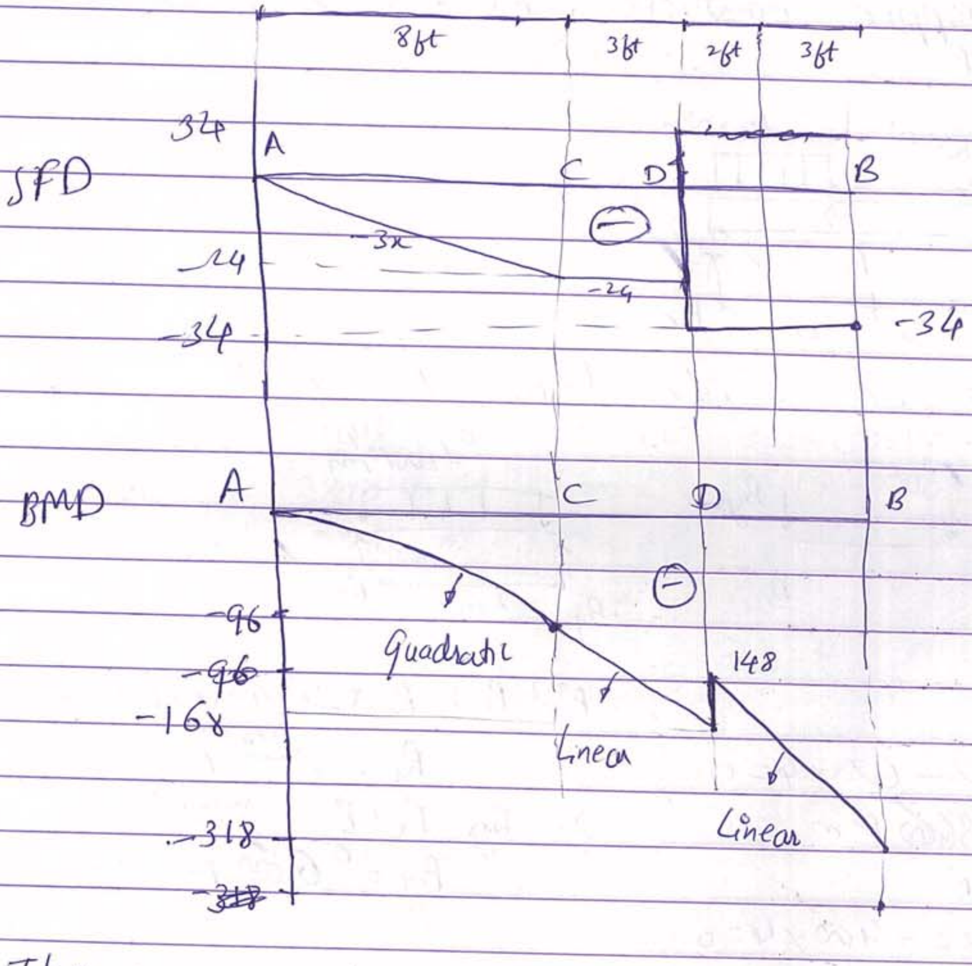


$R_B = 34$ Kips
$M_B = 318$ Kips.ft

$V = -34$ Kips

$-M_B - M - Vx = 0 \qquad M = -Vx - M_B$

$M = -318 + 34x$

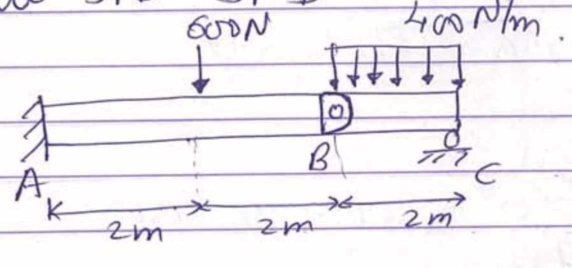


$M = 34x - 318$
@ $x = 5$
$M = -148$

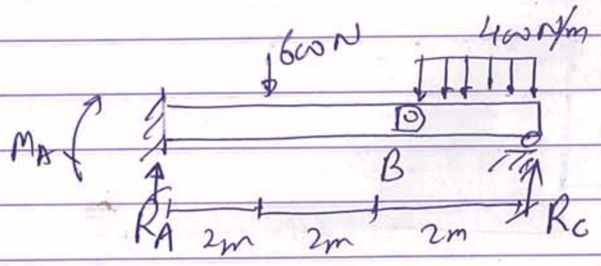
The jump is from −168 to −148 which is due to external moment of 20 Kips.ft.

(29) Problem 5 (Part A).

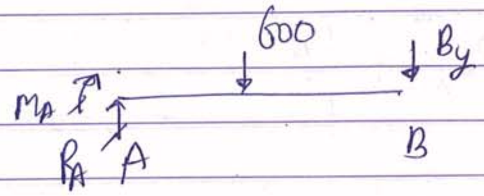
Draw SFD & BMD

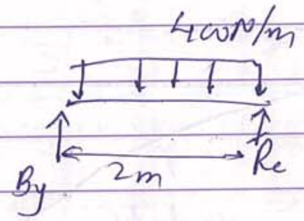


This is a propped cantilever. We have an internal pin at B.



At pin we can split beam into 2 parts.





So,

$-M_A - 600\times 2 - 600\times 4 = 0$

$M_A = -3600\ Nm$

$R_A = 1200N$

$-M_A - 600\times 2 - 400\times 4 = 0$

$M_A = -2800\ Nm$

$R_A = 1000\ N$.

$M@B \Rightarrow R_c\times 2 = 400\times 2\times 1$

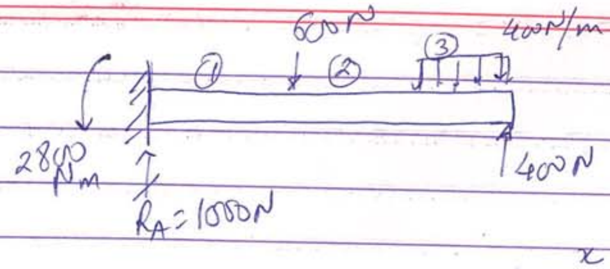
$R_c = \cancel{200}\ 400\ N$

$\Rightarrow R_c + B_y = 800$

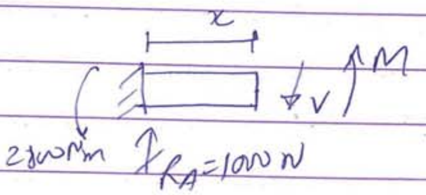
$B_y = \cancel{600\ N}\ 400N$

(30) Problem 5 (Part B)

Now, we will draw SFD & BMD.

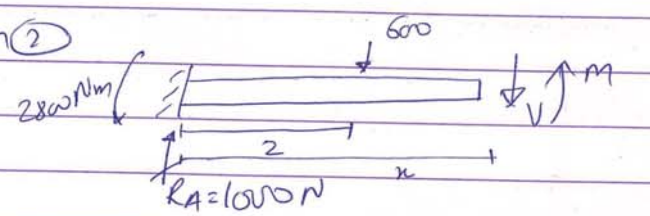


Section ①



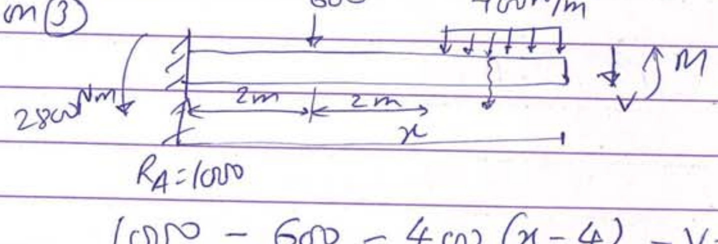
$$V = R_A = 1000\ N$$
$$M + 2800 - 1000x = 0$$
$$M = 1000x - 2800$$
$@\ x = 0 \quad M = -2800 \qquad @\ x = 2 \quad M = -800\ Nm$

Section ②



$$1000 - 600 - V = 0 \qquad V = 400\ N$$
$$2800 + M - 1000x + 600(x-2) = 0$$
$$2800 + M - 1000x + 600x - 1200 = 0$$
$$M = 400x - 1600$$
$@\ x = 2 \quad M = -800 \qquad @\ x = 4 \quad M = 0$

Section ③



$200(x-4)^2$
$200(x^2 + 16 - 8x)$
$200x^2 + 3200 - 1600x$

$$1000 - 600 - 400(x-4) - V = 0$$
$$V = 400 - 400x + 1600$$
$$V = 2000 - 400x$$
$@\ x = 4 \quad V = 400 \qquad @\ x = 6 \quad V = -400$

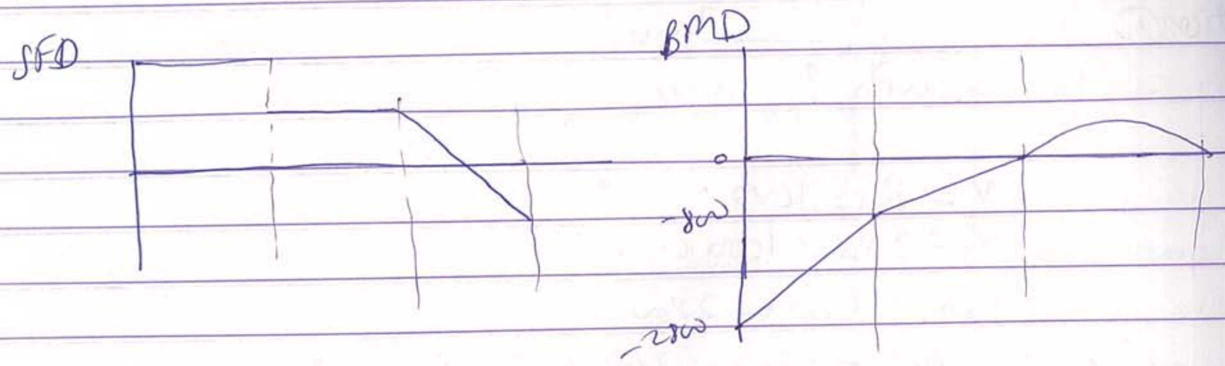
$$2800 + M - 1000x + 600(\cancel{2+x}\ x-2) + 400(x-4)\frac{(x-4)}{2} = 0$$

$$2800 + M - 1000x - 1200 + 600x + 200x^2 + 3200 - 1600x = 0$$
$$4800 + M - 2000x + 200x^2 = 0.$$

$$M = -200x^2 + 2000x - 4800$$

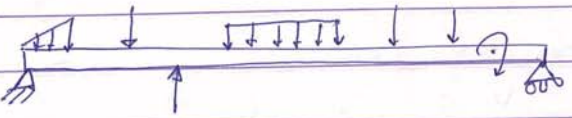
@ $x = 4$ $M = 0$ @ $x = 6$ $M = 0$

$$\frac{dM}{dx} = -400x + 2000 = 0$$
$$x = 5.$$



(31) Issues with Method of sections

Suppose we have a beam with loads -

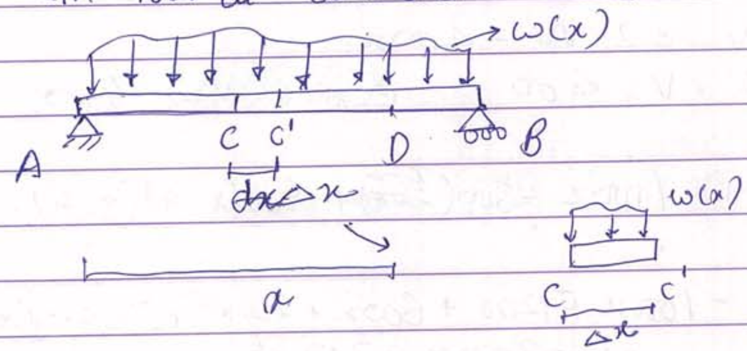


For this type we will need many type of sections
So, this process is tedious.
So, alternate to this is the technique that use relation b/w load, shear force & bending moment.

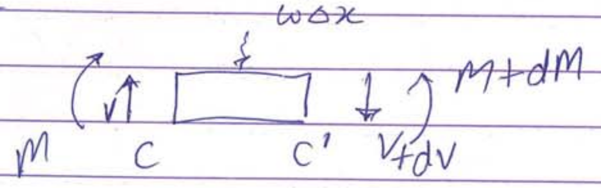
(32) Method using ω-V-M relations

Relationship : $\omega \leftrightarrow V \leftrightarrow M$.

First we will look at distributed load.



If we take small enough section the load can be treated as rectangular load. So, $w\Delta x$ acts at center



Due to $w\Delta x$, shear changes to $V+dV$ & moment changes to $M+dM$.

Relationship #1

$$V - w\Delta x - (V+dV) = 0$$

$$\Rightarrow \frac{\Delta V}{\Delta x} = -w$$

as $\Delta x \to 0$

$$\Rightarrow \boxed{\frac{dV}{dx} = -w}$$

($\because$ Magnitude of change of shear is equal to the load at that point)

$$\int_{x_c}^{x_D} \frac{dV}{dx} = -\int_{x_c}^{x_D} w \Rightarrow \boxed{V_D - V_C = -\int_{x_c}^{x_D} w\,dx}$$

($\because$ Difference of shear b/w two points is equal to negative of area of loading diagram b/w those two points)

Relationship #2

$$M + dM - M - V\Delta x + \underbrace{w\Delta x \frac{\Delta x}{2}}_{\Rightarrow 0} = 0$$

$$\boxed{\frac{dM}{dx} = V} \text{ as } \Delta x \to 0$$

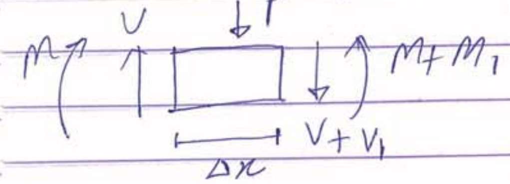
$$\left(\because \int_{x_c}^{x_D} dM = \int_{x_c}^{x_D} V\,dx\right.$$

$$\boxed{M_D - M_C = \int_{x_c}^{x_D} V\,dx}$$

$\therefore$ Difference in moment b/w two points is equal to area of SFD b/w those two points.

Next two relationship is for point load & concentrated moment that we have.

Relationship #3 (Concentrated Load)



$V-(V+V_1)-P=0$

$\boxed{\Rightarrow V_1=-P}$

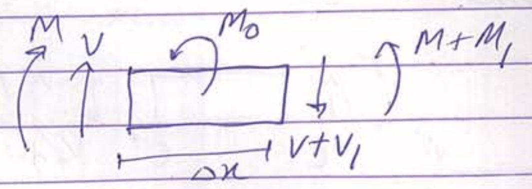
The magnitude of shear change all of a sudden

$M+M_1-M+\frac{P\Delta x}{2}-V\Delta x=0$

$\Rightarrow M_1=V\Delta x-\frac{P\Delta x}{2}$

Δx is very small, M does not go an appreciable change

Relationship #4 (Concentrated Moment)



$V_1=0$
(no change in shear)

$M+M_1-M-V\Delta x+M_0=0$

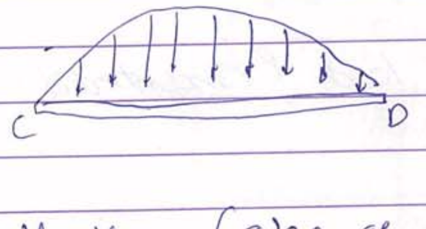
$\boxed{M_1=-M_0}$

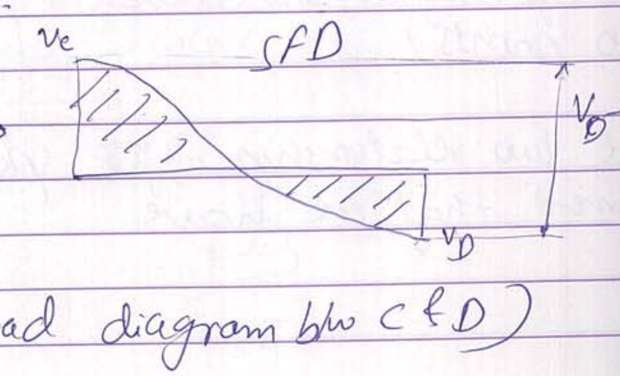
- No change in shear
- Moment changes by appreciable quantity.

(33) Relation Summary

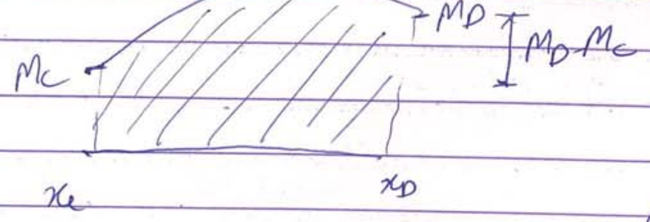
$\frac{dV}{dx}=-\omega(x)$ Slope of SFD is -ve of distributed load $\omega(x)$

Load acting down so, $-\omega(x)$, if load acts up it will be $\omega(x)$.



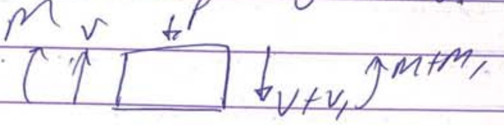


$V_D-V_C=-$(area of load diagram b/w C & D)

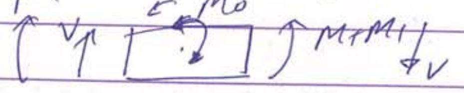


$M_D - M_C = +$ (area of SFD b/w C & D)

$\frac{dM}{dx} = V$ → slope of BMD is equal to the shear force V.

For concentrated load 

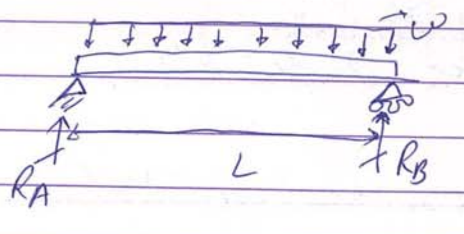
$V_1 = P$, not much change in M.

For concentrated moment 

$M_1 = -M_0$, $V_1 = 0$

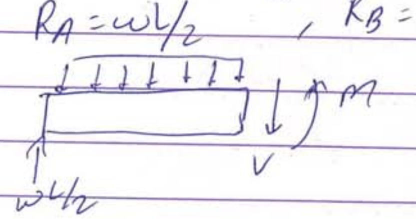
(34) <u>Problem 1</u>

Draw SFD & BMD



We solved using method of sections.

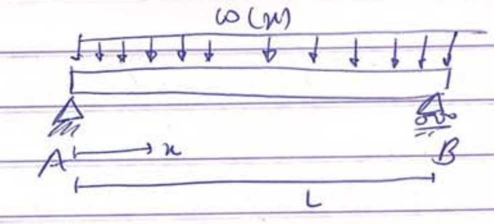
$R_A = wL/2$, $R_B = wL/2$



$\frac{wL}{2} - wx - V = 0$

$V = \frac{wL}{2} - wx$

$M - \frac{wL}{2}x + \frac{wx^2}{2} = 0 \Rightarrow M = \frac{-w}{2}(x^2 - lx)$



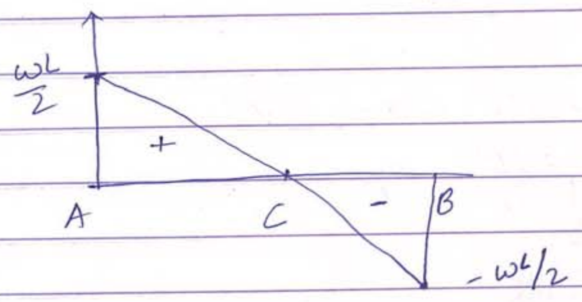
① $\frac{dV}{dx} = -w(x)$

$V_D - V_C = -\int_{x_C}^{x_D} w(x)\,dx$

$R_A = \frac{wL}{2}$, $R_B = \frac{wL}{2}$

② $\frac{dM}{dx} = V$

$M_D - M_C = \int_{x_C}^{x_D} V\,dx$



$\frac{dV}{dx} = -w$

$V_B - V_A = \int_{x_A}^{x_B} -w\,dx = -wL$

$V_B = V_A - wL = \frac{-wL}{2}$,

Let point of zero shear be C. We can find C as →

$$V_C - V_A = -\int_0^{x_C} w\,dx$$

$$0 - \frac{wL}{2} = -w x_C \Rightarrow x_C = L/2$$

Now, $\frac{dM}{dx} = V$, V is linear, so M is quadratic

At A & B M is 0. So, it has a max. Point where $V=0$, then it will be maxima/minima.

At A, $V = wL/2$, so at A slope will be +ve.

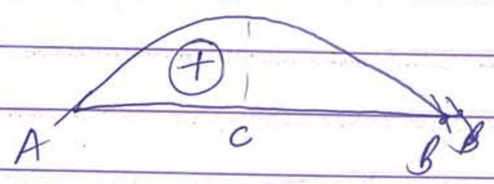
At B, $V = -wL/2$, so at B slope will be -ve.

At C, M will be maximum

$$M_C - M_A = \int_{x_A}^{x_C} V\,dx = \text{Area of shear force diagram}$$

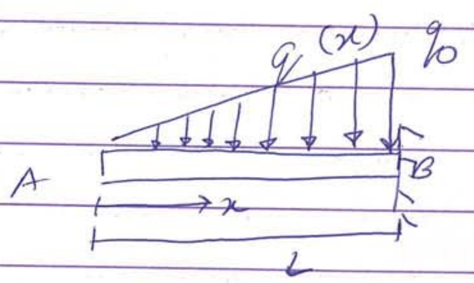
$$M_C - M_A = \frac{1}{2} \times \frac{wL}{2} \times \frac{L}{2} = \frac{wL^2}{8}$$

$$M_C = wL^2/8$$



BMD.

(35) Problem



$q(x) = \frac{q_0 x}{L}$

$$\frac{dV}{dx} = -w(x) = -\frac{q_0 x}{L}$$

So, V will be quadratic.

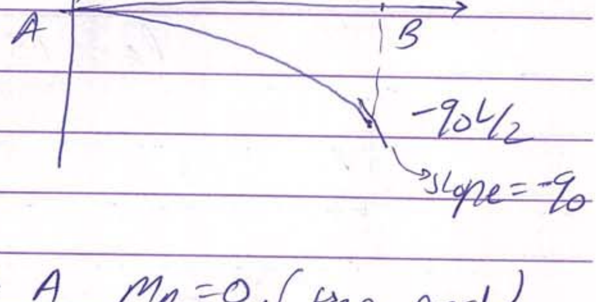
$$\Rightarrow V = \frac{-q_0 x^2}{2L}$$

At A, free end so shear force must be 0.

Slope at A, $\left.\frac{dV}{dx}\right|_{x=0} = \left.\frac{-q_0 x}{L}\right|_0 = 0.$

$$V_B - V_A = -(\text{area of loading diagram})$$
$$= -\frac{1}{2} \times q_0 L$$

$$V_B = -\frac{1}{2} q_0 L$$

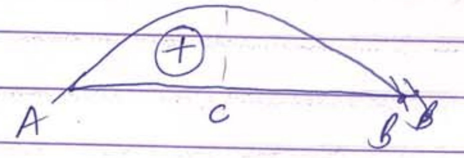


Slope at B $\Rightarrow -\frac{q_0 L}{L} = -q_0$.

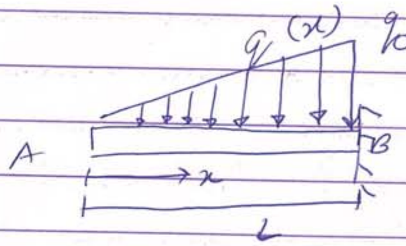
Now for BMD. $\frac{dM}{dx} = V$. At A, $M_A = 0$. (free end)

$$\frac{dM}{dx} = \frac{-q_0 x^2}{2L} \quad \text{So, } M \text{ will be cubic}$$

$$M_B - M_A = \int_0^L -\frac{q_0 x^2}{2L}\, dx = -\frac{q_0 L^3}{6L} = -\frac{q_0 L^2}{6}$$



(35) Problem



$q(x) = \frac{q_0 x}{L}$

$$\frac{dV}{dx} = -w(x) = -\frac{q_0 x}{L}$$

So, V will be quadratic.

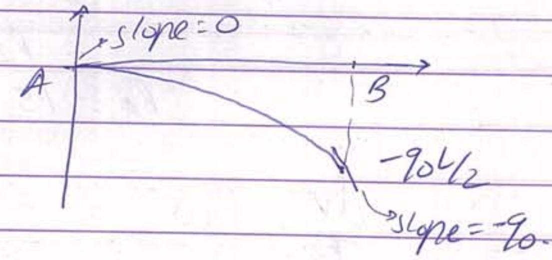
$$\Rightarrow V = \frac{-q_0 x^2}{2L}$$

At A, free end so shear force must be 0.

Slope at A, $\left.\frac{dV}{dx}\right|_{x=0} = \left.\frac{-q_0 x}{L}\right|_0 = 0.$

$$V_B - V_A = -(\text{area of loading diagram})$$
$$= -\frac{1}{2} \times q_0 L$$

$$V_B = -\frac{1}{2} q_0 L$$



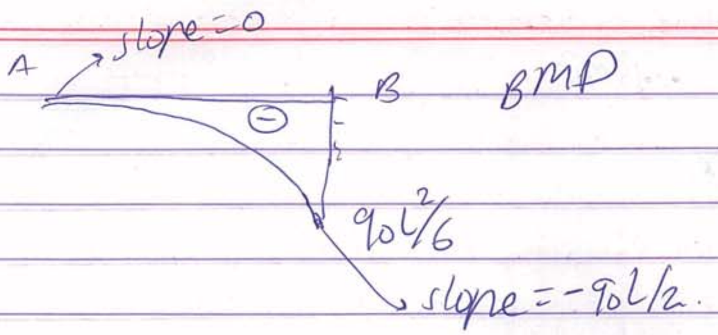
Slope at B $\Rightarrow -\frac{q_0 L}{L} = -q_0$.

Now for BMD. $\frac{dM}{dx} = V$. At A, $M_A = 0$. (free end)

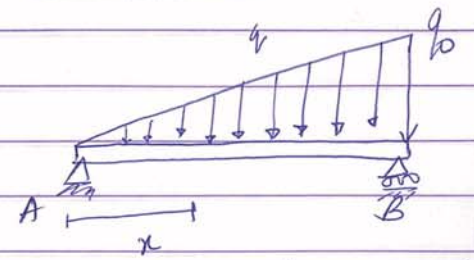
$$\frac{dM}{dx} = \frac{-q_0 x^2}{2L} \quad \text{So, } M \text{ will be cubic}$$

$$M_B - M_A = \int_{x_A=0}^{x_B=L} -\frac{q_0 x^2}{2L}\,dx = -\frac{q_0 L^3}{6L} = \frac{-q_0 L^2}{6}$$

$M_B = \frac{-q_0 L^2}{6}$, $M_A = 0$, slope is negative at B, & at A, slope is 0.



(36) <u>Problem 3</u>



This is a simply supported beam.

$q_0 = 2\ kN/m \qquad L = 4.5m \qquad q = \frac{2}{4.5}x$

$R_A + R_B = q_0 \times \frac{1}{2} \times L = 4.5 \times 2 \times \frac{1}{2} = 4.5$

$R_A + R_B = 4.5\ kN$

$\circlearrowleft M_A = 0$; Force acts at $\frac{2}{3} \times L = 3m$ value $4.5\ kN$.

$\Rightarrow \quad -3 \times 4.5 + R_B \times 4.5 = 0$

$R_B = 3\ kN$

$R_A = 1.5\ kN$.

$\frac{dV}{dx} = -w = -\frac{2}{4.5}x$

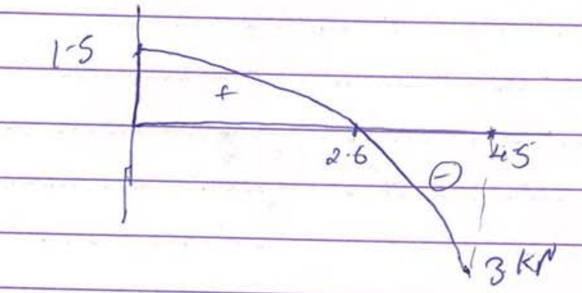
$V_x - V_A = \int_0^x -\frac{2}{4.5} x\, dx$

$V_x - V_A = -\frac{2}{4.5}\frac{x^2}{2} = -\frac{x^2}{4.5}$

$V_A = 1.5\ kN \quad \Rightarrow \quad V_x = 1.5 - \frac{x^2}{4.5}$

@ $x = 4.5 \qquad V_x = -3\ kN$.

$V_x = 0$ @ $1.5 - \frac{x^2}{4.5} = 0 \quad x = 2.6\,m.$



$$\frac{dM}{dx} = V = 1.5 - \frac{x^2}{4.5}$$

$$dM = \left(1.5 - \frac{x^2}{4.5}\right)dx$$

$$M_x - M_A = 1.5x - \frac{x^3}{3 \times 4.5}$$

$$M_x = 1.5x - \frac{x^3}{13.5}$$

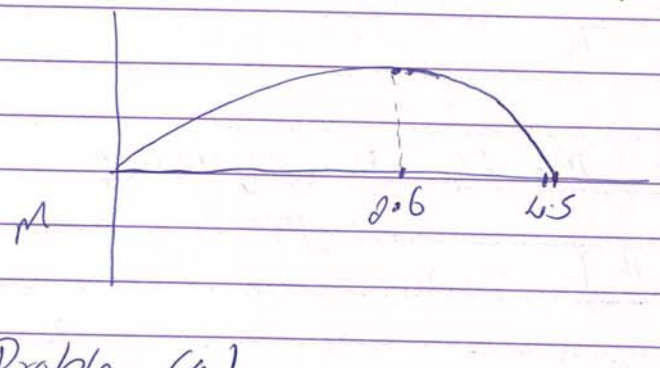
@ $x = 0 \quad M_0 = 0$ @ $x = 4.5 \quad M_{4.5} = 0.$

M_x is max at $\frac{dM}{dx} = 0$

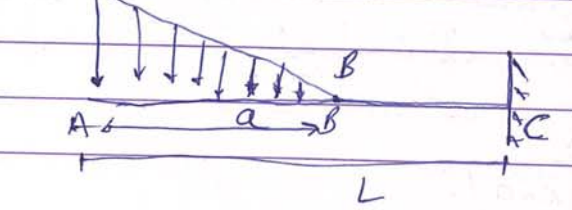
$1.5 - \frac{x^2}{4.5} = 0 \qquad x^2 = 1.5 \times 4.5 \Rightarrow x = 2.6\,m.$

$M_{2.6} = (2.6) \times \left(1.5 - \frac{6.75}{13.5}\right) = 2.6\,kNm$

$M_{max} = 2.6\,kNm.$



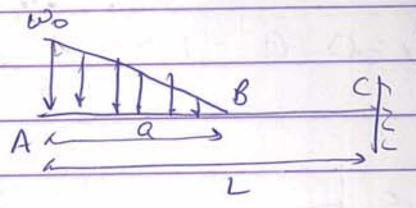
(37) Problem (4)



$R_c - \frac{w_0 a}{2} = 0$

$R_c = \frac{w_0 a}{2}$

$$M_c + \frac{w_0 a}{2} \times \left(l + \frac{2}{3}a\right) = 0$$



$$M_c = \frac{-w_0 a}{2}\left(l + \frac{2}{3}a\right)$$

At A, shear force is 0.

$$\frac{-w_0\left(a - \frac{x}{a}\right)}{a}$$

$$\frac{dV}{dx} = -w(x) = \frac{-w_0 x}{a}$$

$V_B - V_A = -$ (area of loading diagram)

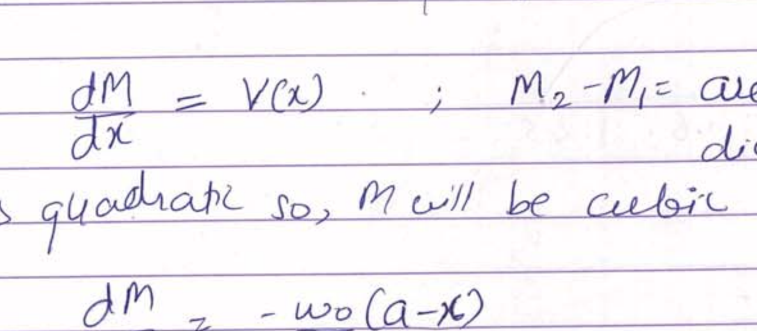
$$V_B - V_A = -\frac{1}{2} \times w_0 \times a$$

$$V_A = 0 \qquad V_B = -\frac{w_0 a}{2}.$$

From A to B, we have quadratic variation

At A, slope is $-w_0$.
At B, slope is 0.

SFD



$$\frac{dM}{dx} = V(x) \quad ; \quad M_2 - M_1 = \text{area of shear force diagram}$$

V is quadratic so, M will be cubic

$$\frac{dM}{dx} = -\frac{w_0}{a}(a - x)$$

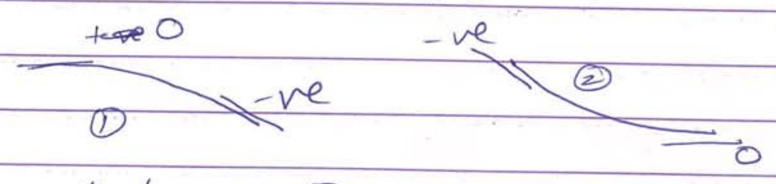
$$dM = \int \frac{w_0}{a}(x-a)\, dx \qquad M = \frac{w_0 x^2}{2a} - w_0 x$$

at $x = 0$ $\quad M = 0$.
at $x = a$ $\quad M = -w_0 a/2$

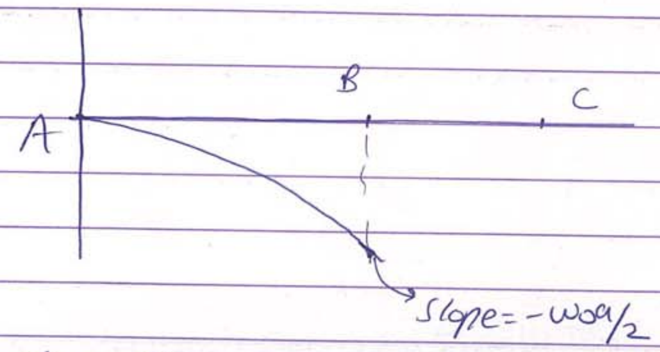
~~Slope of~~ dM/dx equals value of shear force

At A, $V=0$. so, $\frac{dM}{dx}=0$ at A.

At B, $V=-ve$ so, $\frac{dM}{dx}=-ve$ at B.



It will be of ① kind.



~~$M_B - M_A =$~~

$M_B - M_A =$ area of SFD from A to B

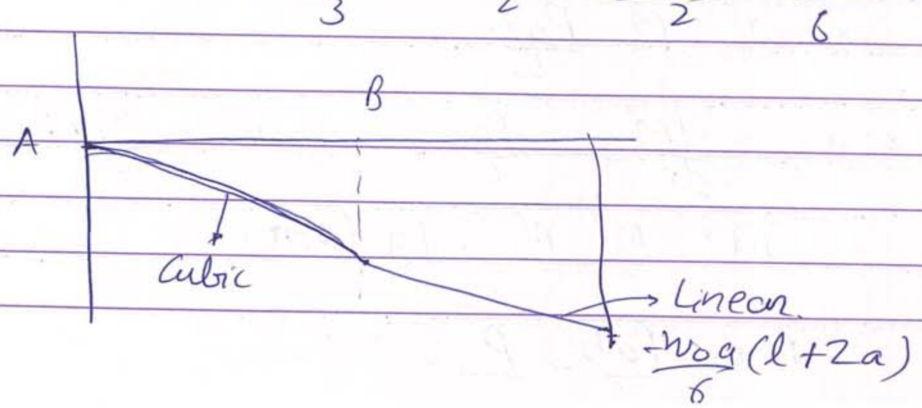
$= \frac{2}{3}\left(\frac{-w_0 a}{2} \times a\right) = \frac{-w_0 a^2}{3}$ (approximately)

From B to C, shear force is same. So moment will linear.

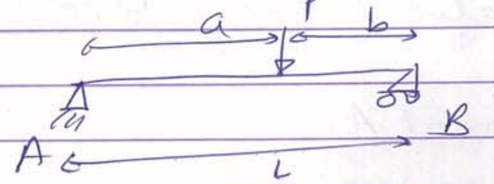
$$M_C - M_B = -\left(\frac{w_0 a}{2} \times (l-a)\right)$$

(area of SFD from B to C)

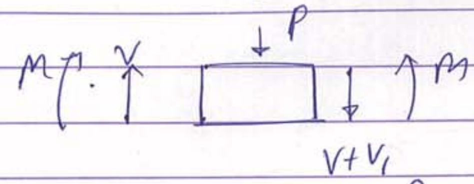
$$M_C = \frac{-w_0 a^2}{3} - \frac{w_0 a l}{2} + \frac{w_0 a^2}{2} = \frac{-w_0 a(3L-a)}{6}$$



(38) Problem 5



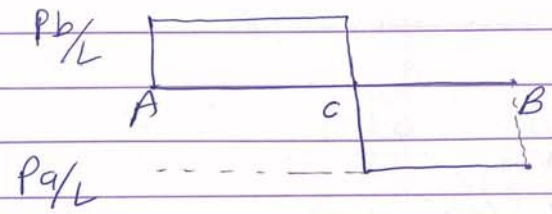
$R_A = \frac{Pb}{L}$, $R_B = \frac{Pa}{L}$



$V - V - V_1 - P = 0 \quad V_1 = -P$

$M = 0.$

SFD



At C, Shear is $V + V_1 = \frac{Pb}{L} - P = -\frac{Pa}{L}$.

At A & B, 0 bending moment.

$\frac{dM}{dx} = V \quad V = \text{Constant}$

So, M will be linear lines.

Till C $\quad \frac{dM}{dx} = \frac{Pb}{L}$, $M = \frac{Pb}{L}x$

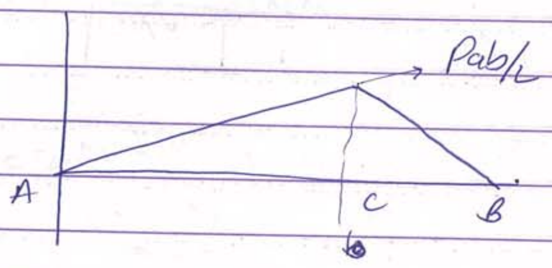
at $x = a \quad M = \frac{Pab}{L}$.

After $x = a$. $\quad \frac{dM}{dx} = -\frac{Pa}{L}$; $M_x - M_c = \frac{-Pa}{L}(b)$

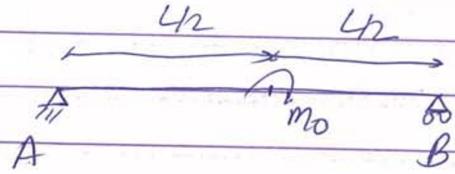
$M_x - M_c = -\frac{Pa}{L}(x-a)$

$M_x = \frac{Pab}{L} - \frac{Pa}{L}(x-a)$

@ $x = L$, $M = 0$, @ $x = a$ $M = \frac{Pab}{L}$



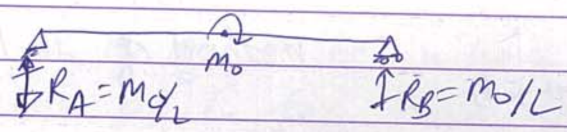
(39) Problem 6.

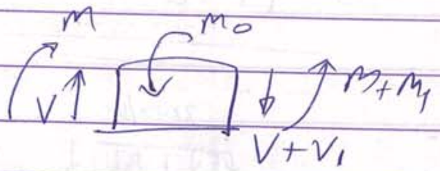


$R_A + R_B = 0$ $\qquad R_A \times L + M_0 = 0$

$R_A = -M_0/L$

$R_B = M_0/L$

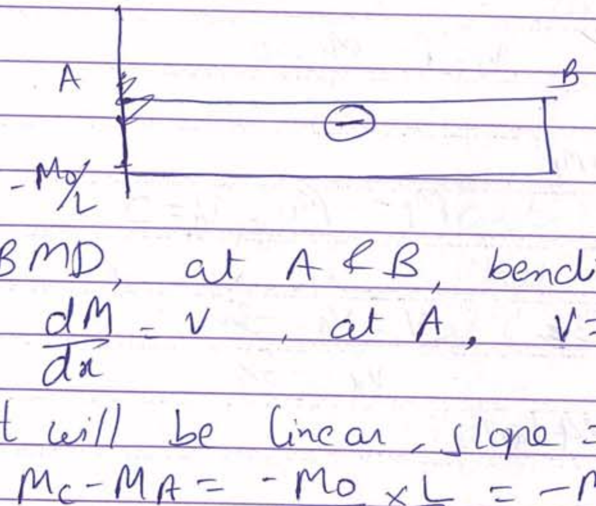




$M_1 = -M_0$
$V_1 = 0.$

SFD. $\frac{dV}{dx} = -w(x) = 0$

$V = \text{constant} = -M_0/L$



for BMD, at A & B, bending moment is 0.

$\frac{dM}{dx} = V$, at A, $V = -M_0/L$

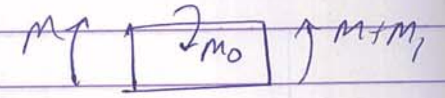
So, it will be linear, slope $= -M_0/L$

$M_C - M_A = \frac{-M_0}{L} \times \frac{L}{2} = -\frac{M_0}{2}$

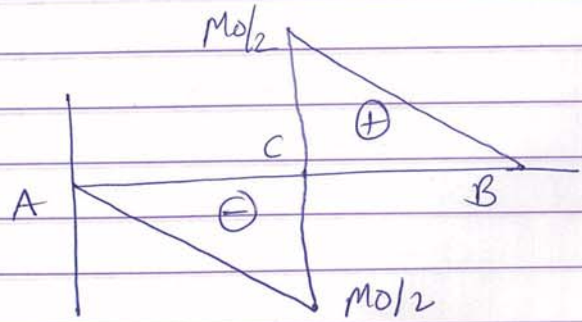
$M_C = 0 - M_0/2 = -M_0/2.$

Just to right of C. $(M+M_1)$

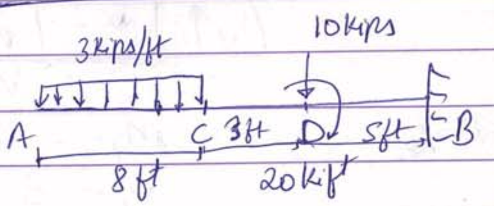
$$M+M_1 = -\frac{M_0}{2} + M_0$$
$$= M_0/2.$$



$$-M + M+M_1 - M_0 = 0$$
$$\underline{\underline{M_1 = M_0}}$$



40 (Problem 7).



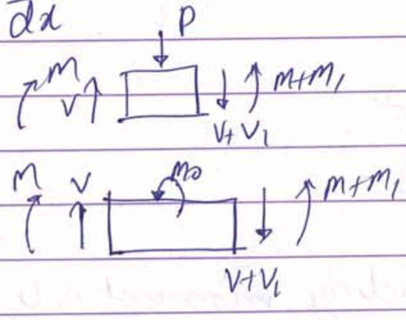
we solved it using method of sections.

① $\frac{dV}{dx} = -\omega(x)$ $\quad V_B - V_A = -$(area of loading diagram)

② $\frac{dM}{dx} = V$ $\quad M_B - M_A =$ area of SFD from A to B.

③ $V_1 = P$, $M_1 \approx 0$

④ $M_1 = -M_0$, $V_1 = 0$.



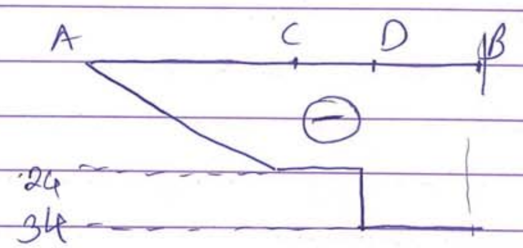
SFD

$V_A = 0$. $\quad \frac{dV}{dx} = -3$ $\quad V_x - V_A = -3x$
$$V_x = -3x.$$
$$V_c = -24 \text{ kips}.$$

From C to D, V remains same at -24 kips

At D, we have a point load so, we go from -24 to -34.
Now, due to ~~shear~~ moment there is no change in shear

so, it remains at -34. From D to B, there is no concentrated load or point load, so shear force remains same.



Now, we can draw BMD.

A to C

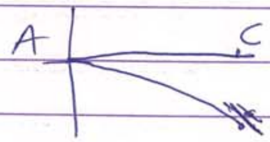
$$\frac{dM}{dx} = V \quad , \quad @ A,\ M_A = 0.$$

slope at A ~~of~~ will be value of shear at A.
So, slope at A for moment is 0.
A to C → V is linear ⇒ M will be quadratic

Slope at C $= -24$.

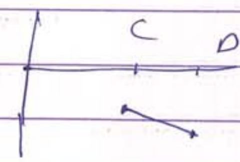
$$M_C - M_A = \text{Area of shear force diagram}$$
$$= \frac{1}{2} \times 8 \times (-24) = -96$$

$$M_C = -96.$$



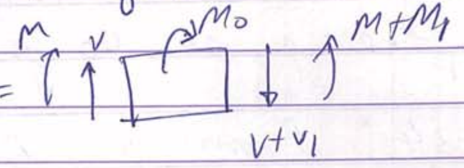
From C to D. Shear force is constant, so, And our starting slope is same that at C so, it is linear with slope equals slope at C.

$$M_D - M_C = -24 \times 3$$
$$M_D = -96 - 72 = -168$$
(just to left of D)



At D we have concentrated moment, so, moment jumps changes just to right of D.

$M_D^{\oplus} =$



$$M + M_0 = M + M_1$$
$$M + M_1 = M + M_0$$
$$M_D^{\oplus} = M_D^{\ominus} + M_0$$

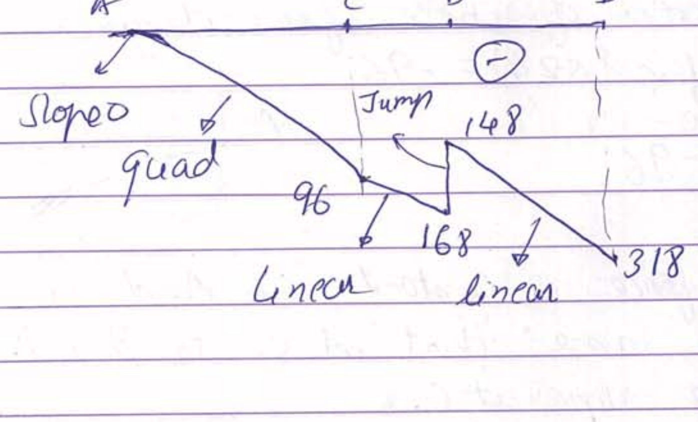
$$M_D^{\oplus} = -168 + 20$$
$$= -148$$

After that D to B.

$M_B - M_D =$ Area enclosed by shearforce diagram

$$M_B = -148 + (-34) \times 5$$
$$= -148 - 170$$
$$= \cancel{-348}\ -318.$$

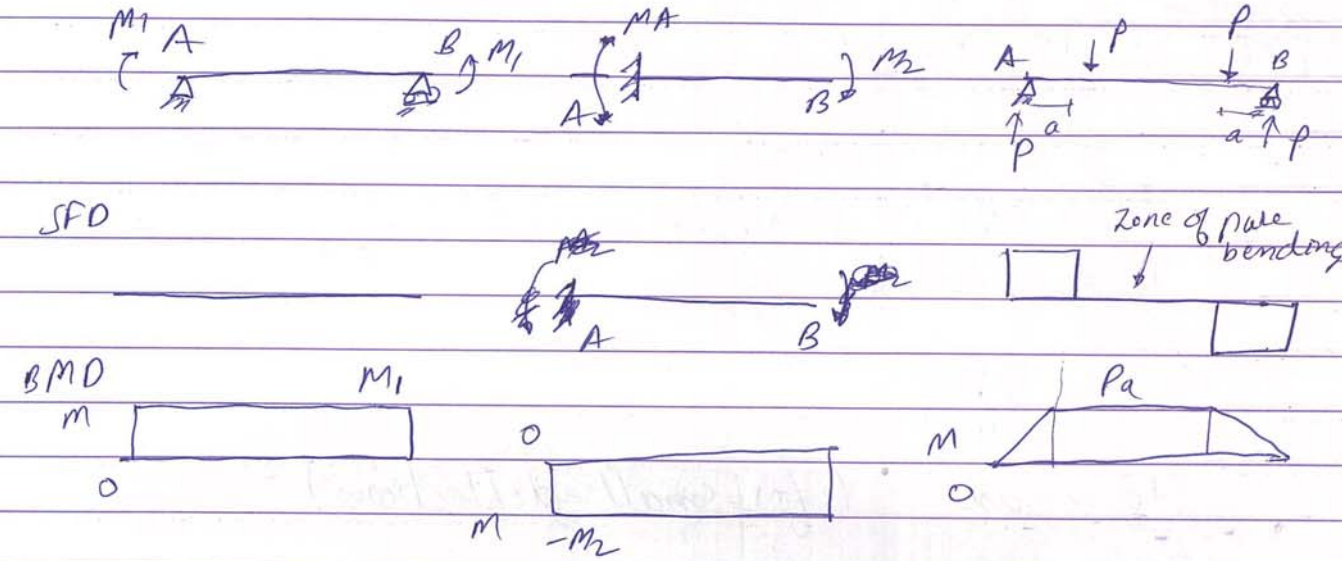
And we have linear variation as shear force is constant. & Slope $= -34$. (value of shear)



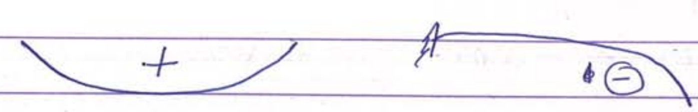
(41) Intro to Pure Bending

We are removing effects of shear & focusing of effects of bending or pure bending.

Defn:- Bending or flexure of beams under constant bending moment.

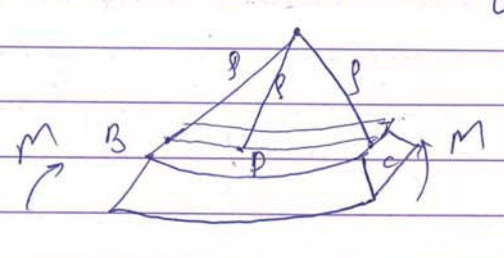


Beam bends

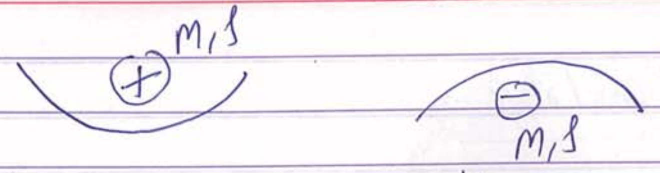


- Internal Bending moment (M) bends the beam & intermediate sections into the arc of a circle. (All derivations are valid for prismatic bars)

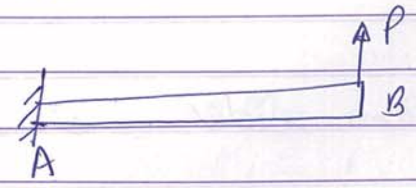
We will have radius of curvature

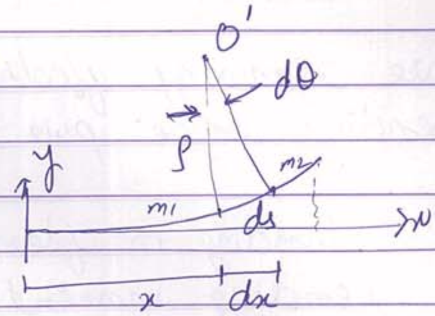


At B, C & D radius of curvature is ρ.



Sagging Hogging





Bends in a arc of circle.
For small deflections

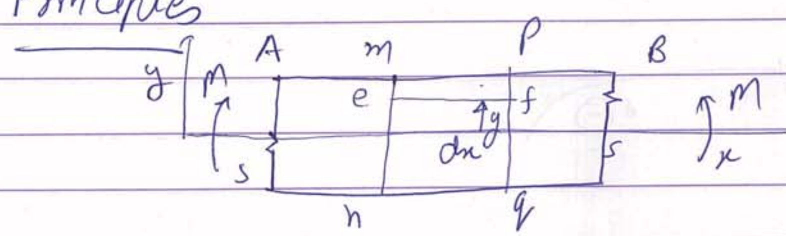
$$ds = \rho\, d\theta$$

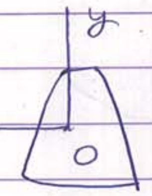
ρ = radius of curvature
k = curvature = $1/\rho$

$$k = \frac{d\theta}{ds}$$

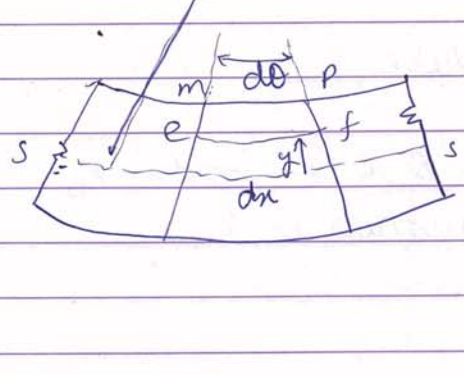
$ds \sim dx$ (for small deflections).

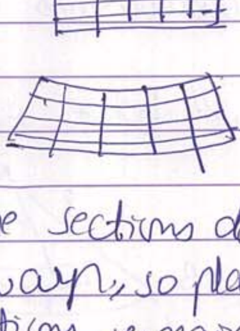
Principles





prismatic bar





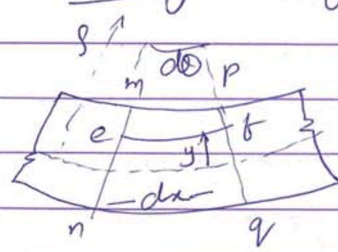
① The sections don't warp, so plane sections remain plane before & after bending

② Top line contracts] in sagging.
Bottom line expands]

③ One line does not change at all. (Neutral axis) (NA).
(No tension, nor compression).

The above 3 are assumptions (mostly facts).

(42) <u>Deriving the flexure formula</u>. (Part A)



original length of ef = $dx = \rho\, d\theta$
Deformed length of ef = $(\rho - y)\, d\theta$

Longitudinal Strain = $\dfrac{\text{change in length}}{\text{original length}}$

$= \dfrac{-y\, d\theta}{\rho\, d\theta} = -y/\rho$

$1/\rho = \kappa$ = curvature

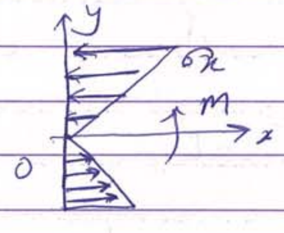
$$\boxed{\varepsilon_x = -y/\rho = -\kappa y}$$

$\sigma = E\varepsilon$ (from Hooke's Law)

$$\boxed{\sigma_x = -E\varepsilon_x = -E\kappa y}$$

So, σ is a linear fn of y. (σ, ε varies linearly with distance from neutral Axis)

$y = 0 \Rightarrow \varepsilon_x = 0 \quad \sigma_x = 0$. (At Neutral Axis).



(Q3) Deriving of Flexure Formula (Part B)

We will use two fundamental equations:

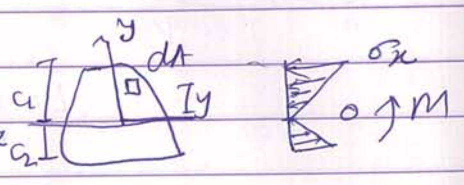
① First eqⁿ of statics → Force eqⁿ

② Second eqⁿ of statics → Moment eqⁿ.

The neutral axis lies about centroid axis

Force eqⁿ:

$$\int_A \sigma_x \, dA = -\int_A E k y \, dA = 0$$



$$\Rightarrow \int y \, dA = 0$$

First moment of area w.r.t. z axis is 0.

∴ z axis must pass through the centroidal axis

∵ z axis is neutral axis ∴ neutral axis must pass through the centroidal axis.

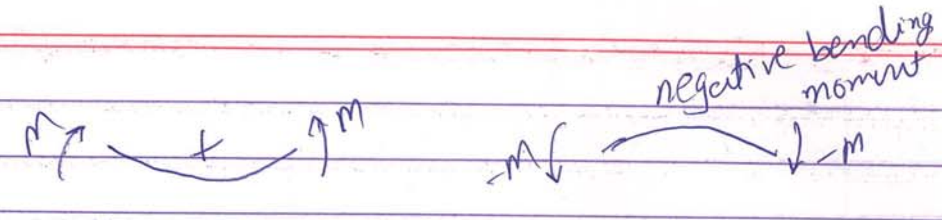
Moment eqⁿ:

$$M = -\int_A \sigma_x y \, dA = \int_A kEy^2 dA = kE\int y^2 dA$$

$$M = kEI \quad \left(I = \int y^2 dA\right) = I_{zz} \text{ (about z axis)}$$

Area moment of Inertia. about z.

$$\boxed{\frac{M}{EI} = \frac{1}{\rho} = k}$$



Positive curvature — -ve curvature

$$\boxed{k = \frac{1}{\rho} = \frac{M}{EI}}$$

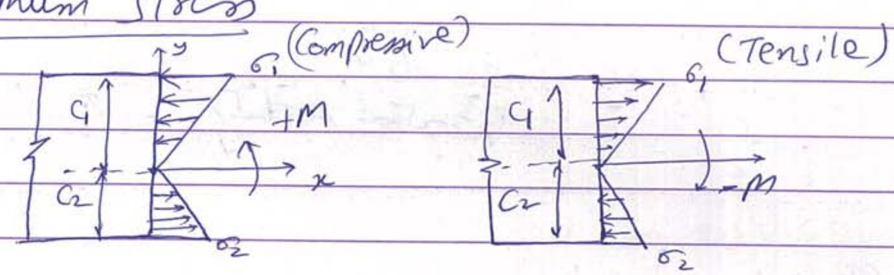
$\boxed{EI \Rightarrow \text{Flexural Rigidity}}$

If M is +ve, k is +ve; M is -ve; k is -ve.

$$\sigma_x = -kEy \quad ; \quad k = \frac{M}{EI}$$

$$\Rightarrow \boxed{\sigma_x = -\frac{My}{I}} \quad \text{flexure formula.}$$

Maximum stress



$$\boxed{\sigma_1 = -\frac{Mc_1}{I} = \frac{-M}{S_1} \qquad \sigma_2 = \frac{Mc_2}{I} = \frac{M}{S_2}}$$

$S_1 = \frac{I}{c_1}$: $S_2 = \frac{I}{c_2}$ are known as section module

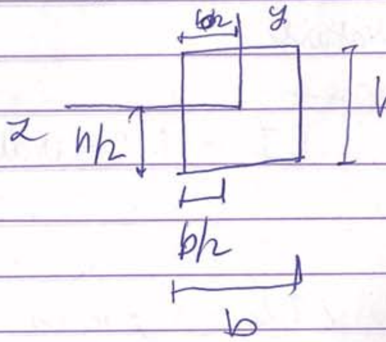
(These are usually provided in designer's handbook)

$$|\sigma_x| = \left|\frac{My}{I}\right|$$

+ve for tensile & -ve for compressive.

Doubly Symmetric Sections

Rectangular section.

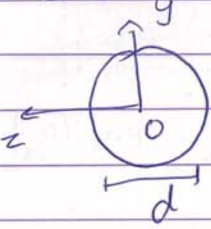


$I = \frac{1}{12} bh^3$

$\underline{\underline{I_{zz}}}$

$$S_1 = S_2 = \frac{I}{c} = \frac{I}{h/2} = \frac{bh^2}{6}$$

Circular section



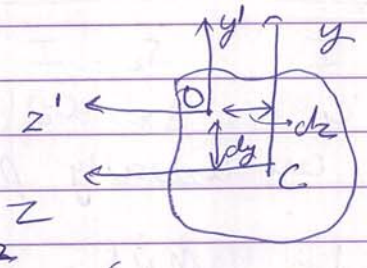
$$I = \frac{\pi d^4}{64}$$

$$S_1 = S_2 = \cancel{\pi d} \frac{I}{d/2} = \frac{\pi d^3}{32}$$

Review (Moment of Inertia)

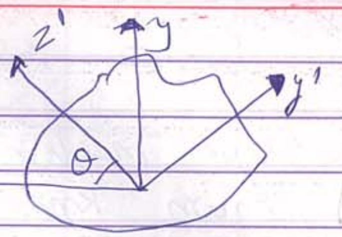
$I_z = \int_A y^2 dA \quad I_y = \int_A z^2 dA \quad I_{yz} = I_{zy} = \int yz\, dA$

(about centroid C)



$I_{z'} = I_z + A d_y^2 \;;\; I_{y'} = I_y + A d_z^2$ (about point O)

$$\underbrace{I_{y'z'} = I_{yz} + A d_y d_z}_{\text{Parallel Axis Theorem}}$$



Rotation of Axis

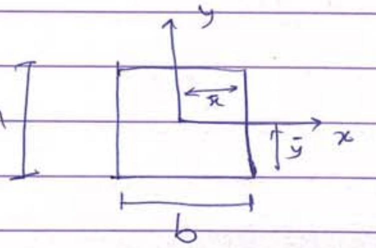
$$I_{z'} = I_y \sin^2\theta + I_z \cos^2\theta - I_{yz} \frac{\sin 2\theta}{2}$$

$$I_{y'} = I_y \cos^2\theta + I_z \sin^2\theta + I_{yz} \frac{\sin 2\theta}{2}$$

$$I_{y'z'} = (I_z - I_y) \frac{\sin 2\theta}{2} + I_{yz} \cos 2\theta$$

For principal axes ($I_{y'z'} = 0$) : $\tan 2\theta = \dfrac{2 I_{yz}}{I_{yy} - I_{zz}}$

For plane areas

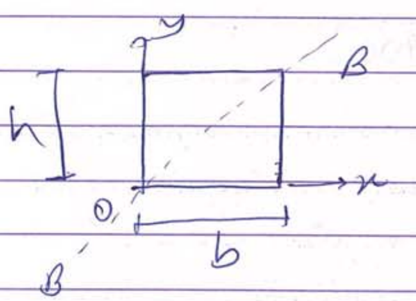


$A = bh \quad \bar{x} = b/2 \quad \bar{y} = h/2$

$I_x = \dfrac{bh^3}{12} \quad I_y = \dfrac{hb^3}{12} \quad I_{xy} = 0$

Polar moment of inertia about an axis passing through the centroid & perpendicular to plane of the rectangle is the sum of the moments of inertia about the two perpendicular axes in that plane

$$J = I_x + I_y = \frac{bh^3}{12} + \frac{b^3h}{12} = \frac{bh}{12}(b^2 + h^2)$$



$I_x = \dfrac{bh^3}{3} \quad I_y = \dfrac{hb^3}{3} \quad I_{xy} = \dfrac{b^2h^2}{4}$

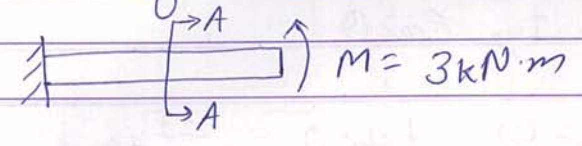
$$I_p = \frac{bh}{3}(h^2 + b^2)$$

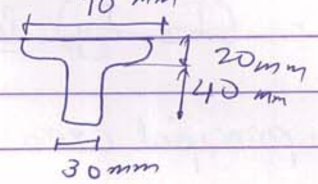
$$I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

Similarly we can get for others. We can use designer's handbook for I, S (moment of inertia and section moduli)

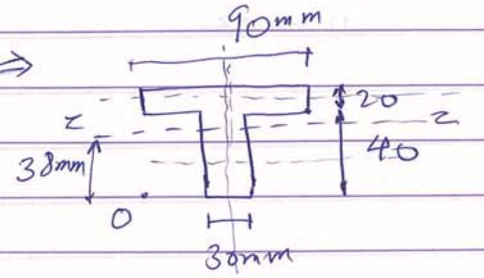
(44) <u>Problem I</u>

- A cast-iron machine part is acted upon by 3 kN·m couple shown. Knowing that $E = 165$ GPa & neglecting the effect of fillets, determine
(a) max tensile and compressive stresses in the casting
(b) radius of curvature





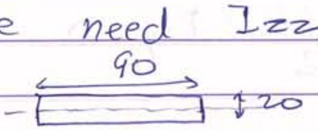
$\Rightarrow$



$$y_c = \frac{40\times30\times20 + 90\times20\times(40+10)}{40\times30 + 90\times20}$$

$$= \frac{114000}{1200+1800} = 38\text{ mm}$$

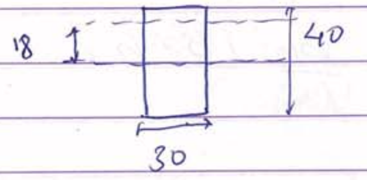
We need I_{zz}.



$$I_{zz_1} = \frac{1}{12}\times 90\times(20)^3 + 90\times20\times(2+10)^2$$

$$= 60,000 + 259200$$

$$= 319200$$



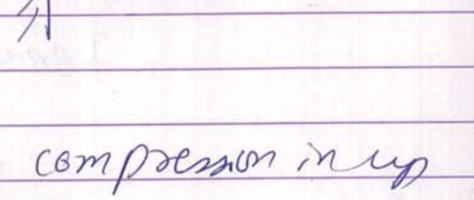
$$I_{zz_2} = \frac{1}{12}\times 30\times(40)^3 + 30\times40\times(18)^2$$

$$= 160000 + 388800$$

$$= 548800$$

$$I_{zz} = I_{zz_1} + I_{zz_2}$$

$$= 868,000$$



$$|\sigma_x| = \left|\frac{M\,y}{I}\right|$$

Looking at moment, it will be compression in up & tension in down.

$$\cancel{|\sigma_{max}|} = \sigma_{c_{max}} = -\frac{3\times1000\times(20+2)\times1000}{868000}$$
$$= -76 \text{ MPa}$$

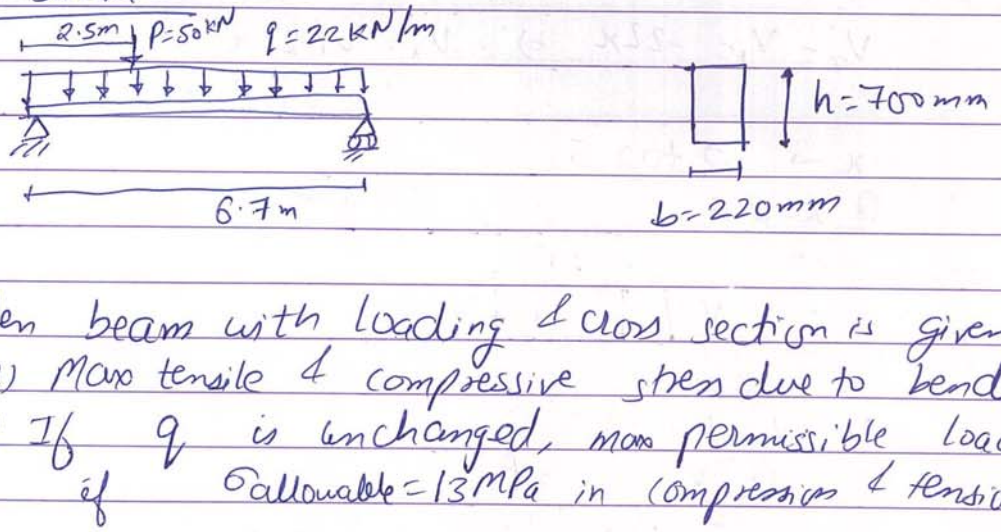
$$\sigma_{t,max} = \frac{3\times1000\times(38)\times1000}{868000}$$
$$= 131.34 \text{ MPa}$$

(b) Radius of curvature.

$$k = \frac{1}{\rho} = \frac{M}{EI}$$

$$\rho = \frac{EI}{M} = \frac{165\times10^9 \times 868000}{3\times10^6\times10^6}$$
$$= 47,740 \text{ mm}$$
$$= 47.74 \text{ m}$$

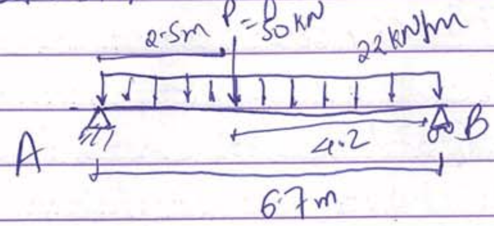
(45) Problem 2



Wooden beam with loading & cross section is given. Find (a) Max tensile & compressive stress due to bending
(b) If q is unchanged, max permissible load P, if $\sigma_{allowable} = 13$ MPa in compression & tension

⇒ Centroid will be at 350 mm & 110 mm.
Here bending is not uniform. But at that small location/ chunk ~~can be approximated~~ moment moment is constant so the bending formula/ flexure formula still holds good.

Here we will find SFD & BMD. We can also use relationship to find SFD & BMD



$R_A + R_B = 50 + 22 \times 6.7$
$= 197.4$

$\curvearrowright M_A = 0 \quad -50 \times 2.5 - 6.7 \times 22 \times \frac{6.7}{2} + R_B \times 6.7 = 0$

$R_B = \frac{50 \times 2.5}{6.7} + \frac{22 \times 6.7}{2} = 92.36$

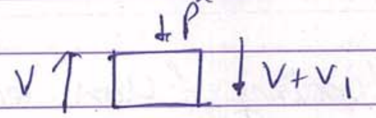
$R_A = 105.04$

$\frac{dV}{dx} = -w(x) = -22$

$V_x - V_1 = -22x \Rightarrow V_x = V_1 - 22x$
$= 105.04 - 22x$

$x \rightarrow 0$ to 2.5.
@ $x = 2.5$ $\quad V_{2.5} = 50.04$

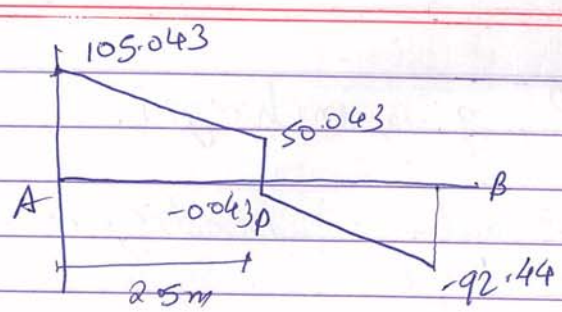
at $x = 2.5^+$



$V + V_1 + P - V = 0$
$V + V_1 = V - P$
$= 50.04 - 50 = -0.04$

After $x > 2.5$
$\frac{dV}{dx} = -w(x) = -22$

$V_x - V_1 = -22x$
$V_x = V - 22x = -0.04 - 22x$
$x = 4.2 \quad V = -92.46.$



$\frac{dM}{dx} = V$, @A , M=0 .

V = +ve , so, slope = +ve. till P.

$M_{@P} - M_{@A}$ = area under curve = +ve

$M_{@P}$ = +ve.

So, curve up.

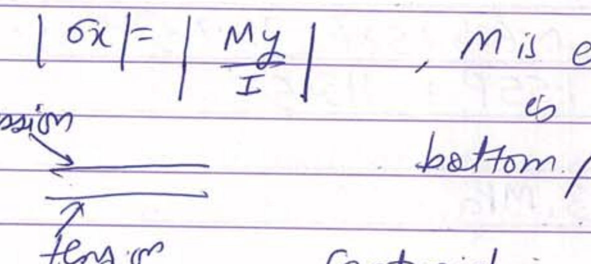
$\frac{dM}{dx} = 105.04 - 22x \Rightarrow M = -11x^2 + 105.04x$

$M_{@x=2.5} = -11 \times (2.5)^2 + 105.04 \times 2.5$

$= 193.85$

At 2.5^+, V is -ve, so, dM/dx is -ve.

So, decrease & @ B M=0. So, it is

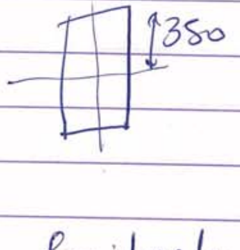


$|\sigma x| = \left| \frac{My}{I} \right|$, M is entirely +ve. So, it is sagging. So bottom part will be in tension.

compression

tension

Centroid is at center.



$\sigma_t = \frac{My}{I} = \frac{193.85 \times 10^6 \times 350}{\frac{1}{12} \times 220 \times (700)^3}$

$= 10.8$ MPa.

Similarly $\sigma_c = -10.8$ MPa at top fibre.

(b) We need to find P, q is unchanged.

We need to find adequate value of bending moment.
Due to P, R_A also changes.

$$R_A + R_B = P + 22 \times 6.7$$
$$R_A + R_B = P + 147.4$$

$$M_{@B} = 0 \quad -R_A \times 6.7 + P \times 4.2 + 22 \times 6.7 \times \frac{6.7}{2} = 0$$

$$R_A = 0.62P + 73.7$$

$$\frac{dM}{dx} = V \quad \frac{dV}{dx} = -w$$

$$V_x = 0.62P + 73.7 - 22x$$

$x = 2.5 \quad V_{2.5} = 0.62P + 18.7$

$$\cancel{\frac{dM}{dx} = V \quad M = 0.62Px + 18.7x}$$

$$\cancel{M_{@2.5} = 0.62 \times 2.5 P + 18.7 \times 2.5}$$
$$\cancel{= 1.55P + 46.75}$$

$$\frac{dM}{dx} = V \quad M_x = 0.62Px + 73.7x - 11x^2$$

$x = 2.5 \quad M = 0.62 \times 2.5 \times P + 73.7 \times 2.5 - 11 \times (2.5)^2$
$$M = 1.55P + 115.5$$

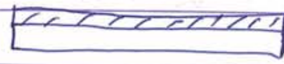
So, given $\sigma = 13$ MPa.

$$13 = \frac{(1.55P + 115.5) \times 10^6 \times 350}{\frac{1}{12} \times 220 \times 700^3}$$

$P = 75.4$ kN.

(46) Bending of composite Beams

Composite beams are beams made of two dissimilar materials, stuck together ~~by a~~ or held together by a mechanism.
Till now we studied for homogenous material.



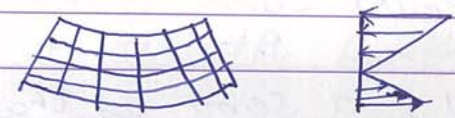
→ attached with glue

Now, we ~~will~~ will see how to calculate stresses for composite beams.
Composite beams are very very important.
eg: Reinforced concrete beam (made of concrete & steel)

For homogeneous section (made of one single material)

$\varepsilon_x = -ky$; $\sigma_x = -Eky$

$|\sigma| = \left|\frac{My}{I}\right|$

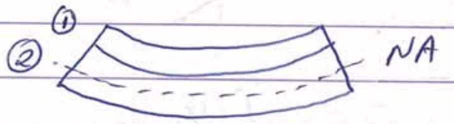


Composite Beam

→ material 1 E_1
→ material 2 E_2

E_1 & E_2 are two different young's modulus.

Bending behaviour

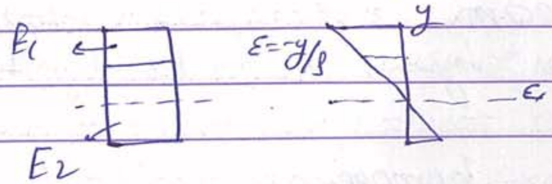


For composite beam neutral axis need not be at the center.

Here also all 3 assumptions (or facts) are valid.
1. Sections don't warp, so plane section remain plane before & after bending.
2. Top line contracts and bottom line expands in ~~sagg~~ sagging.
3. One line does not change at all (Neutral Axis)

So, $\varepsilon = -y/\rho$ is still valid.

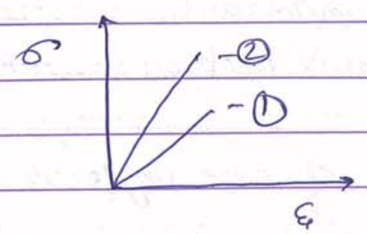
Strains are still proportional.



Stress will not have this linear variation across the section.

$$\sigma = E\varepsilon$$

Stress-strain curve for this two materials

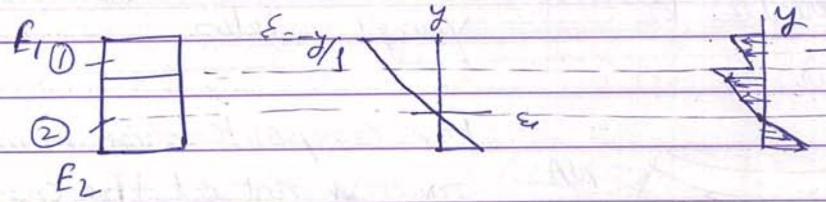


Assuming material 2 is stiffer, $E_2 > E_1$.

We have no-slip condition at join of two materials. So, it bends same in two materials.

So, strain in material 1 is same as that of material 2. For a particular strain value, σ is more for material 2 than material 1.

Our stress looks as→

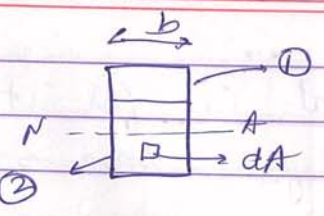


$$\sigma_1 = -\frac{E_1 y}{\rho} \qquad \sigma_2 = -E_2 y/\rho .$$

So, now we have discontinuity for stress.

To calculate stresses, we will use <u>method of transformed stresses</u>

Take the same beam



We take a small element in part ②, area dA.

So, $dF_2 = \sigma_2 \, dA$

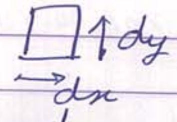
$$\sigma_2 = \frac{-E_2 y}{\rho}$$

$$\Rightarrow dF_2 = -\frac{E_2 y}{\rho} dA$$

Let $\boxed{n = E_2/E_1}$ ratio of modulus of elasticity / <u>modular ratio</u>

So, $E_2 = \cancel{nE_2}\ nE_1$

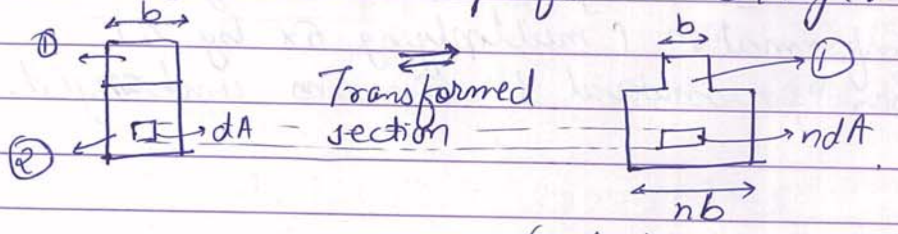
So, $dF_2 = -\frac{E_1 y}{\rho}(n\,dA)$



~~Let~~ Now, in our dA ~~length = dx breadth = dy~~.

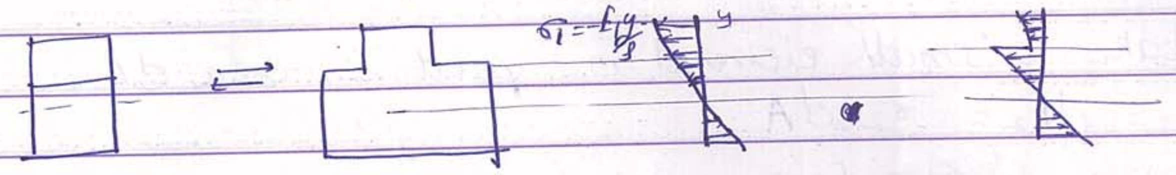
Last expression tells us that, the material 1 can take the same load provided we have increased the area n times of original area. We have transformed material 2 to material 1. To keep dy constant, we can ~~k~~ increase the width ~~b b~~ by n.

Why we don't want to increase height, because we don't want our neutral axis to change. ~~and~~ or our strain profile to change.



(whole is made of material ①)

Strain profile remain same. And now material is made of same material. So, for strain will have linear line ~~and again for stress we will~~ have linear line



Now, to get the stresses for material 2. we multiply by modular ratio.

So, after transformation we are calculating stress for one particular material and then we back transform for original material.

Here E_2 was greater than E_1, so stress for material 2 increased

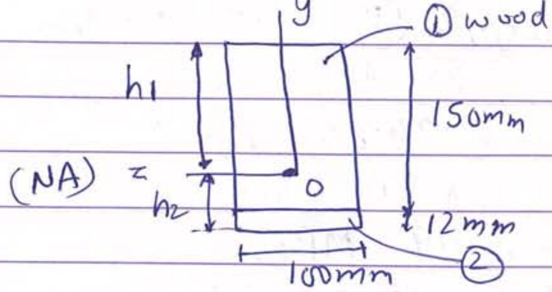
So, $n > 1 \rightarrow$ widening
$n < 1 \rightarrow$ shrinking.

Summary

1) Calculate modular ratio n (E_2/E_1)
2) Create a transformed section by transforming area of material 2 to material 1.
3) Calculate stresses etc. for transformed section consisting of only material 1.
4) Calculate stresses for material 2 by back-transformation (multiplying σ_x by n).
5) Stresses for material 1 remains unchanged.

(47) Problem Example

- Composite beam made of wood (1) & steel plate (2)
- $E_1 = 10.5$ GPa, $E_2 = 210$ GPa
- for a +ve bending moment of 6 kNm:
  - (i) max & min stresses in wood.
  - (ii) max & min stresses in steel.



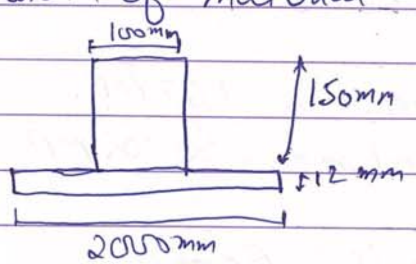
If z is our neutral axis Then in wood, due to the bending moment we will have compression on top and tension in bottom.

And in steel plate we will have tension as it is below neutral axis.

We will transform to one of material.
We transform steel to wood.

n = modular ratio $= E_2/E_1 = 210/10.5 = 20.$

The width of material 2 increases.

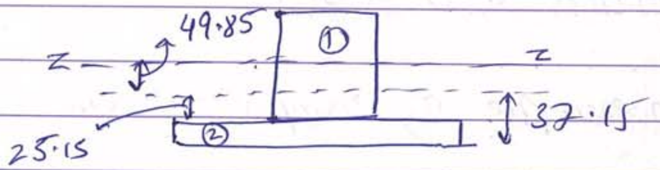


$$\bar{y} = \frac{2000 \times 12 \times 6 + 100 \times 150 \times 87}{2000 \times 12 + 100 \times 150}$$

$$= \frac{2 \times 12 \times 6 + 15 \times 87}{2 \times 12 + 15}$$

$$= \frac{144 + 1305}{24 + 15} = \frac{1449}{39}$$

$$= 37.15 \text{ mm}$$



$$I_{zz_1} = \frac{1}{12} \times 100 \times (150)^3 + 100 \times 150 \times (49.85)^2$$

$$= 28125000 + 37275337.5$$

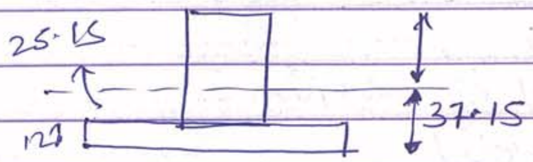
$$= 65400337.5 \text{ mm}^4 = 65.4 \times 10^6 \text{ mm}^4$$

$$I_{zz_2} = \frac{1}{12} \times 2000 \times (12)^3 + 2000 \times 12 \times (25.15)^2$$

$$= 288000 + 23,287,740$$

$$= 23,575,740 \text{ mm}^4 = 23.57 \times 10^6 \text{ mm}^4$$

$$I = \cancel{63}(65.4 + 23.57) \times 10^6 \, mm^4$$
$$= 88.97 \times 10^6 \, mm^4$$



$$|\sigma| = \left|\frac{My}{I}\right|$$

$$\sigma_c = -\frac{My}{I} \quad \text{at top}$$

$$= -\frac{6 \times 10^6 \times (150 + 12 - 37.15) \, N\,mm^2}{88.97 \times 10^6 \, mm^4}$$

$$= -\frac{6 \times 124.85}{88.97} \, N/mm^2$$

$$= -8.419 \, MPa.$$

$$\sigma_{t,wood} = \frac{My}{I} = \frac{6 \times 10^6 \times (37.15 - 12)}{88.97 \times 10^6} \, N/mm^2$$

$$= 1.69 \, MPa$$

At same point $\sigma_{steel} = 20 \times 1.69 \, MPa$
$= 33.8 \, MPa.$

For $\sigma_{steel,max} = \dfrac{6 \times 10^6 \times 37.15 \times 20}{88.97 \times 10^6}$

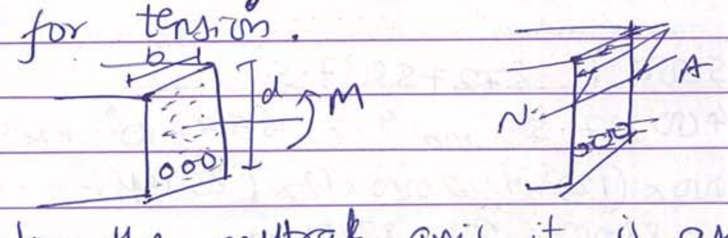
$= 50.106 \, MPa$

$\sigma_{wood,min} = -8.419 \, MPa$ $\quad$ $\sigma_{wood,max} = 1.69 \, MPa$
$\sigma_{steel,min} = 34 \, MPa$ $\quad$ $\sigma_{steel,max} = 50.106 \, MPa.$

Note on Reinforced Concrete (RC) Beams:

Reinforced concrete is an example of composite beam.

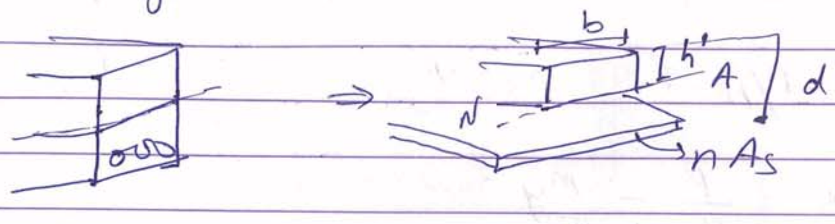
Concrete is very weak in tension so steel is provided ~~to~~ for tension.



Below the neutral axis it is assumed that concrete ~~does not take stress at all.~~

does not contribute at all to taking the stress that might occur in the beam. The entirety of tension is assumed to be taken ~~in~~ by steel.

The transformed section looks as



$n = E_s/E_c$ = Modular Ratio

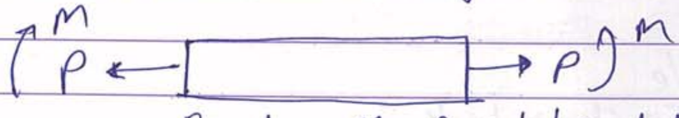
We converted to steel.

~~The method is a bit obsel~~ There was a time when ~~concte~~ reinforced concrete beam was also solved using similar principles of method of transformed sections.

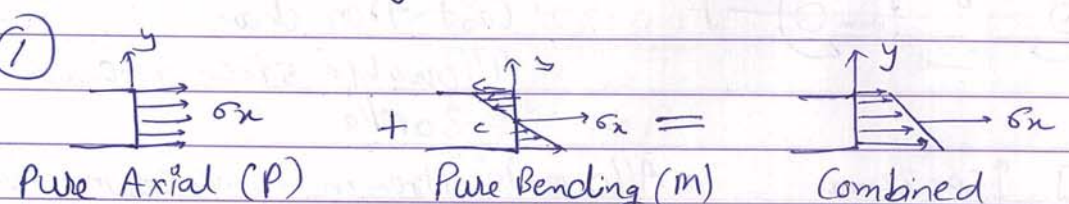
(48) <u>Bending with Axial loading</u>

- Pure bending with axial loading

Here material is homogenous



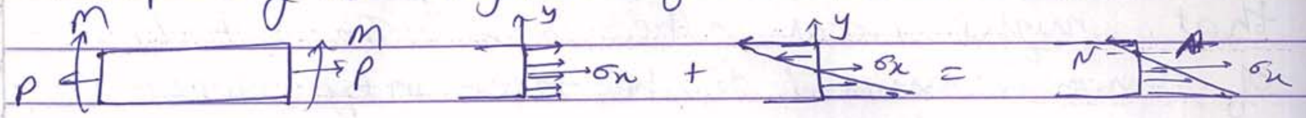
Bending M, Axial Load P



(Stress distribution is uniform)

It can happen that entire section is under tension.

(2) Depending on magnitude of P & M, it can also be:-



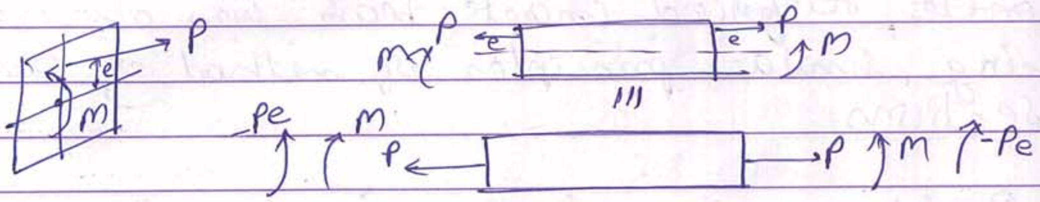
Here in above two examples P acts at center.

for (1) & (2) $\sigma_x = P/A$ & $\sigma_x = \frac{-My}{I}$

Total $\sigma_x = \frac{P}{A} - \frac{My}{I}$

will be +ve for tensile

If P acts at an eccentricity?

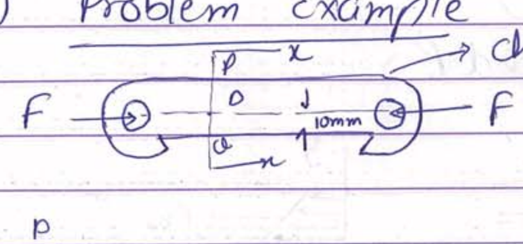


Revised formula =

$$\sigma_x = \frac{P}{A} - \frac{My}{I} + \frac{Pey}{I}$$

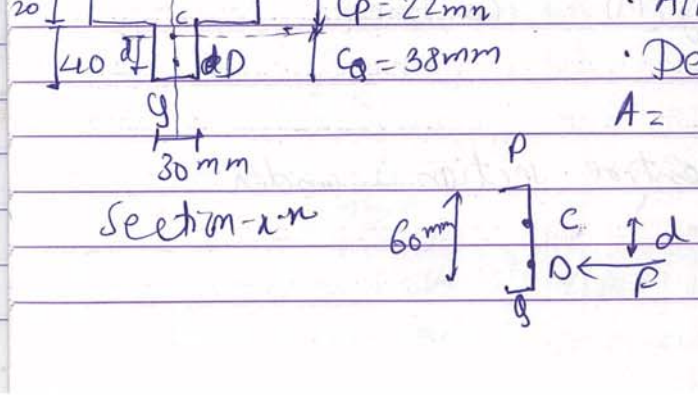
Additional moment.

(49) Problem Example



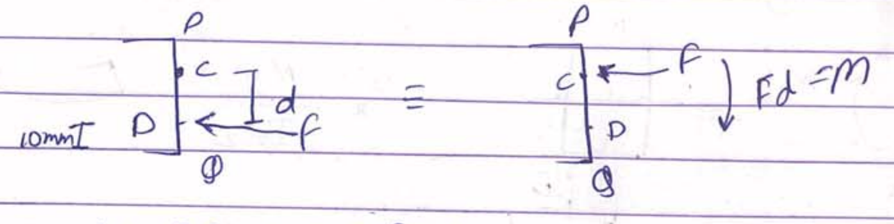
- Cast-iron chain link is shown
- Allowable stress in tension $= 30\,MPa$
- Allowable stress in compression $= 120\,MPa$
- Determine largest force F

$A = 3 \times 10^{-3}\,m^2$, $I = 868 \times 10^{-9}\,m^4$

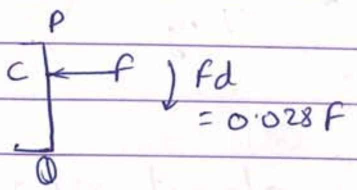


Section-x-x

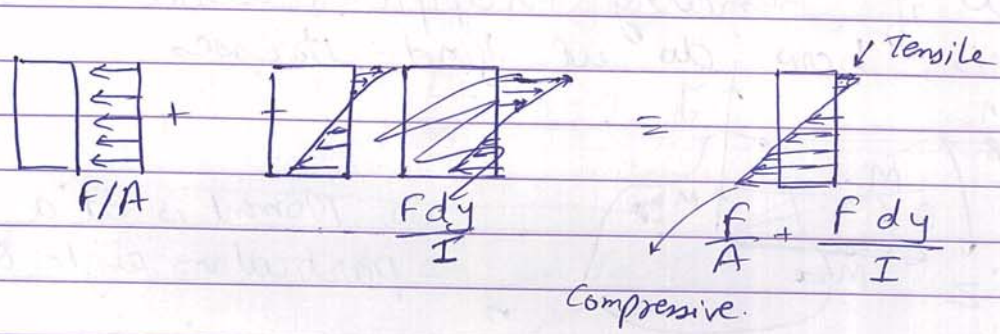
Location of center is given $C_P = 22m$ & $C_Q = 38\,mm$.
C is centroid.
Force acts at a distance d. So, we shift force at centroid.



$d = 38 - 10 = 28mm$



$f \rightarrow$ uniform stress distribution
$Fd \rightarrow$ Bending stress



Overall tensile stress $\sigma_t = \sigma_{t,M} - F/A$

$$= \frac{F \times 0.028 \times 0.022}{868 \times 10^{-9}} - \frac{F}{3 \times 10^{-3}}$$

$$= \cancel{0.007}\ 376.3\ F.$$

$$\sigma_c = \sigma_{c,M} - F/A$$

$$= -\frac{F \times 0.028 \times 0.038}{868 \times 10^{-9}} - \frac{F}{3 \times 10^{-3}}$$

$$= -1559.14 F$$

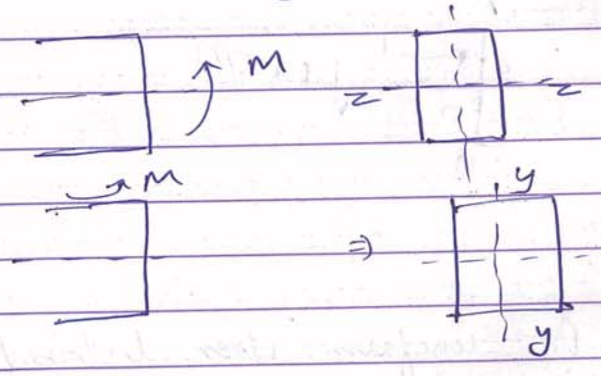
$376.3F = 30 \times 10^6$
$F = 79.7 kN$

$1559.14F = 120 \times 10^6$
$F = 76.9\ kN.$

So, largest force $F = 76.9 kN.$

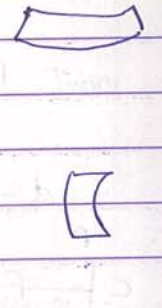
(50) Unsymmetric Bending

Bending moment acting about arbitrary axis is unsymmetric Bending.

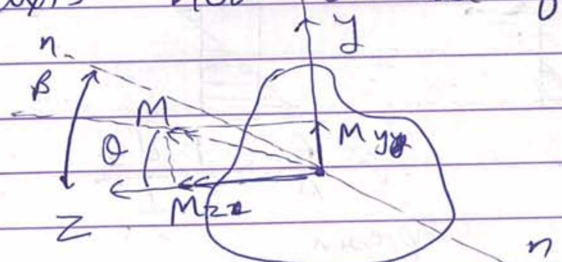


$$\sigma_x = \frac{My}{I_{zz}}$$

$$\sigma_x = \frac{Mz}{I_{yy}}$$



Now if bending happens about arbitrary axis how do we find stresses.

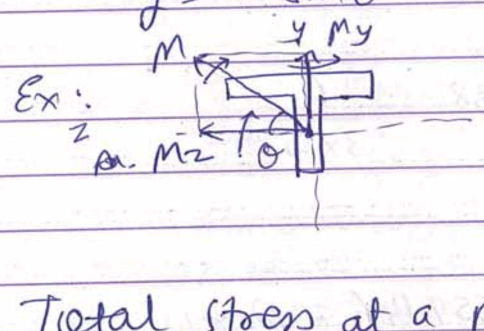


So, Moment is at a particular angle θ.

We resolve the moment in Z & Y axis.
Now, how do we get neutral axis.

$M_y = M\sin\theta \qquad M_z = M\cos\theta$

Ex:



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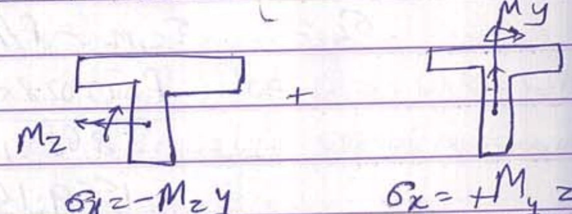


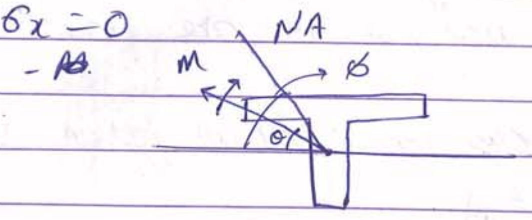
$$\sigma_x = -\frac{M_z y}{I_z} \qquad \sigma_x = +\frac{M_y z}{I_y}$$

Total stress at a particular point

$$= -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

(Don't be too bogged down by signs)

Neutral axis is a line where stress is 0.



$\sigma_x = 0$ about NA $\Rightarrow$ $\dfrac{-M_z y}{I_z} + \dfrac{M_y z}{I_y} = 0$

$$y = \left(\frac{I_z}{I_y}\tan\theta\right) z \qquad (y = mz)$$

straight line of slope $m = (I_z/I_y)\tan\theta$.
= slope of neutral axis.

$$\boxed{\tan\phi = \frac{I_z}{I_y}\tan\theta}$$

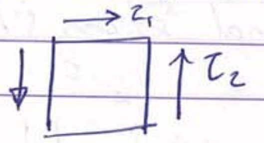
Neutral axis at an angle ϕ.

(51) Introduction to shear in Beams

We saw double shear and single shear previously.
We express shear stress in τ and shear force in V. &
shear force acts along a face.

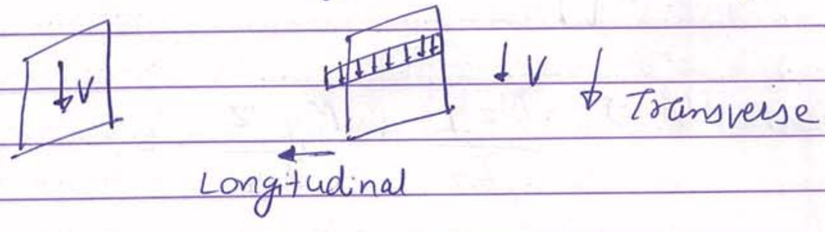
$$\tau_{avg} = \frac{V}{A} = \frac{\text{shear force}}{\text{total area}}$$

We saw complementarity of shear stress i.e. ~~to~~ if shear force acts on a plane, then it gives rise for shear force in other plane/faces just to maintain equilibrium, and forces are same.



$\tau_1 = \tau_2$ & $\tau = G\gamma$.
(shear strain)

- We also saw shear force and bending moment diagram
- In case of pure bending we saw what kind of stresses and strains ~~invo~~ are invoked by M.
- Now, we will focus on shear force V.



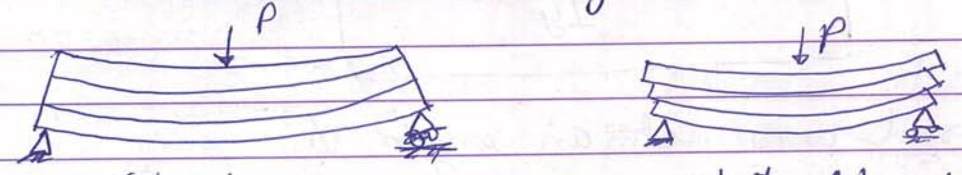
But does V only cause transverse stress on the cross section?

⇒ Logically Yes.

Consider the two cases →

① Boards are glued together

② Boards are not glued together.

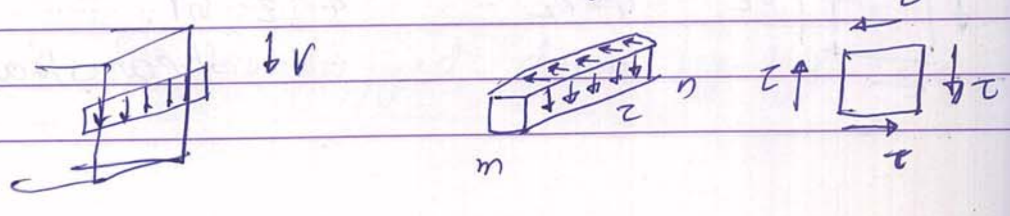


In not glued board, if we apply force P, the boards stick out. So, they slide across one ~~When they~~ another. So, glue resists some kind of force in case 1 & beam bends.

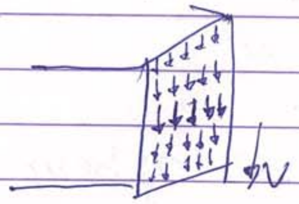
This shows that moment & shear are kind of related.

Glue resists force in longitudinal direction to keep all boards together.

It tells that although we have all forces in transverse direction, you also develop due to <u>complementarity</u>, the longitudinal ~~stress~~ shear.



So, Complementary shears coexists.



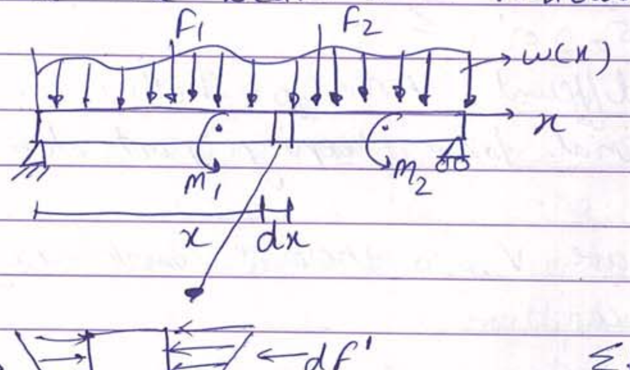
Due to transverse shear, longitudinal shear exist. It coexist

Here the length of arrows are longer ~~#~~ towards the center, than towards the edges. It shows that shear stresses are varying across the depth of cross section.

(52) Deriving the Shear Formula.

~~For~~ We saw that transverse shear stress gives rise to longitudinal shear stress.
~~to~~ Due to τ going down we have a complimentary shear stress in longitudinal dir^n.
We will derive shear formula.

Consider the beam with various sets of loading.



Consider cross section ⇒ dx

(1)

$\Sigma f_x = 0$ satisfied.

So, we have $dF'' = \frac{\sigma M}{y}$

& $dF' = \frac{\sigma(M+dM)}{y}$.

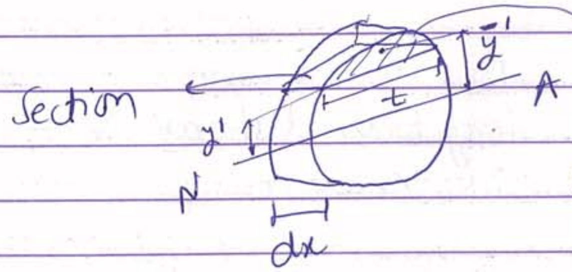
Element equilibrium

& element is in eq^{m}.

~~&~~

}

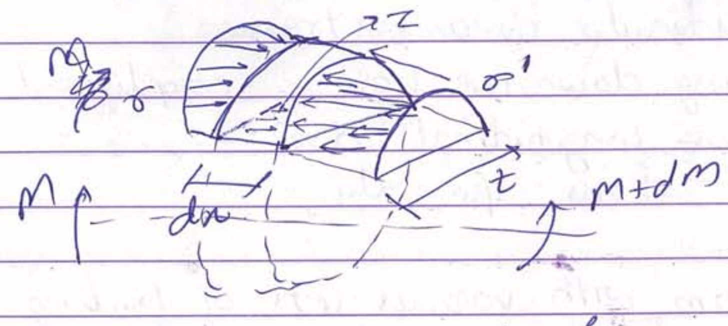
② The cross section for prismatic beam.



Cross section of beam

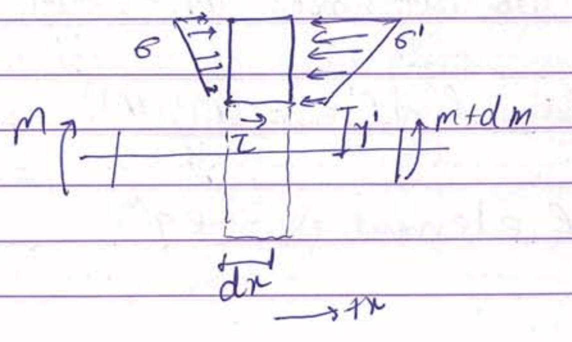
We are looking at small section of a beam.
The section is at a distance y' from NA.
The ~~sec~~ section area is A'.
The centroid of area is at $\bar{y}'$ from NA.
The section is also is in equilibrium.

③ Part element



Stresses acting → σ, σ', & τ
σ & σ' are different. Area for both is same.
So, there is additional force keeping part element in eqm.
Along face we have V, so there is complementary τ in longitudinal face.

④ 2D representation of part element



~~From ③ we get ④~~ n
④ is ③ in 2D
We can write eqn of eqm
$\Sigma F_x = 0$.

$$\int_{A'} \sigma \, dA' + \tau t \, dx - \int_{A'} \sigma' \, dA' = 0$$

$$\sigma = \frac{My}{I} \qquad \sigma' = \frac{(M+dM)}{I} y$$

The shear exist because the moment on both side of faces are different.

$$\int_{A'} \left(\frac{My}{I}\right) dA' + \tau t \, dx - \int_{A'} \left(\frac{(M+dM)}{I} y\right) dA' = 0$$

$$\Rightarrow \qquad \tau t \, dx - \int_{A'} \frac{dM}{I} y \, dA' = 0$$

$$\Rightarrow \qquad \tau t \, dx = \frac{dM}{I} \int_{A'} y \, dA'$$

$$\tau = \frac{dM}{dx} \times \frac{1}{I \times t} \int_{A'} y \, dA'$$

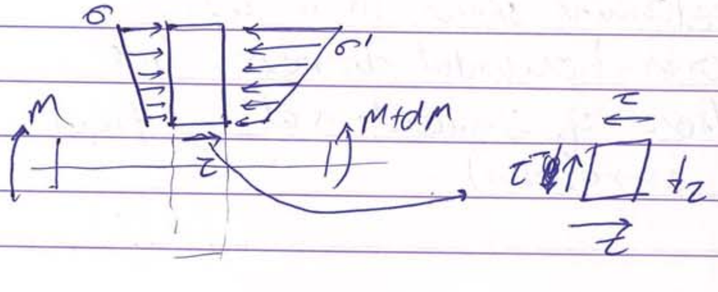
$$\frac{dM}{dx} = V$$

$$\tau = \frac{V}{It} \times \underbrace{\int_{A'} y \, dA'}_{Q}$$

$$\boxed{\tau = \frac{VQ}{It}} \qquad Q = \int_{A'} y \, dA'$$

$V \rightarrow$ applied shear force
$I \rightarrow$ area moment of inertia for entire section.
$t \rightarrow$ thickness of the section
$Q \rightarrow$ Area of chunk multiplied by distance of centroid of area from neutral axis

$$= \bar{y}' A' = \int_{A'} y \, dA'.$$



τ is same along the transverse & longitudinal plane at the same point because of complementarity.

$$\tau = \frac{VQ}{It}$$

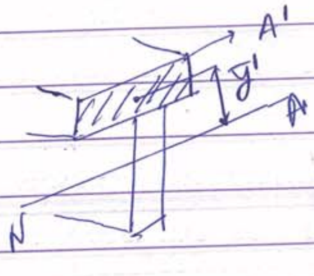
τ = Shear stress in the member at the point located a distance y from the neutral axis. This stress is assumed to be constant and therefore averaged across the width t of the member.

V = the shear force, determined from the method of sections and the equations of eqm

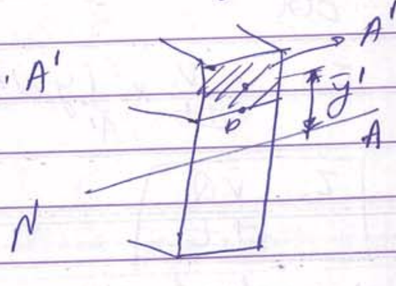
I = moment of inertia of the entire cross-sectional area calculated about the neutral axis.

t = width of the member's cross section, measured at the point where τ is to be determined.

Q = $\bar{y}'A'$, where A' is the area of the top (or bottom) portion of the member's cross section, above (or below) the section plane where t is measured, & $\bar{y}'$ is the distance from the neutral axis to the centroid of A'.

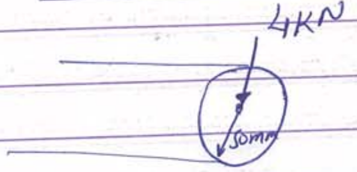


$Q = \bar{y}' \cdot A'$



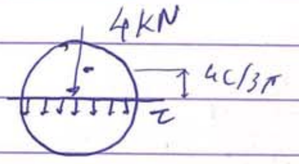
We could also have taken bottom area

(53) Problem 1



- Solid tube with radius $c = 50mm$
- shear at cross section = 4 KN.
- Determine shear stress acting over horizontal diametric line

(Hint: Centroid of semicircle is at $\frac{4c}{3\pi}$ from base. c = radius)



$V = 4 \times 10^3\ N$

$t = 1\cancel{0}0\,mm. = 2c$

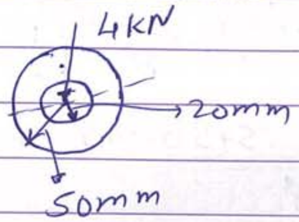
$Q = \bar{y}' \cdot A' = \frac{4c}{3\pi} \times \pi \times c^2$

$I = \frac{\pi c^4}{4}$

$$\tau = \frac{VQ}{It} = \frac{V \times \frac{4}{3} \times c^{3}}{\frac{\pi c^4}{4} \times 2c} = \frac{8\,\cancel{16}\,Vc^2}{3\pi}$$

$$\tau = \frac{8 \times 4 \times 1000 \times (50)^2}{3\pi} = 0.68\ MPa.$$

If we had hollow cylinder.



$\tau = \frac{VQ}{It}$

Q changes, I changes, t changes.

$Q = A \times \bar{y}$; $A = \pi(50^2 - 20^2) = 2100\pi/2$

$= 6597.3\ mm^2.$

$\bar{y} = \frac{4}{3\pi} \frac{(50^3 - 20^3)}{(50^2 - \cancel{2}0^2)} = 23.645mm$

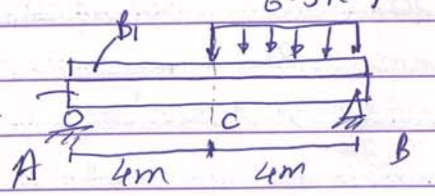
$Q = 155993.1585/2\ mm^3$

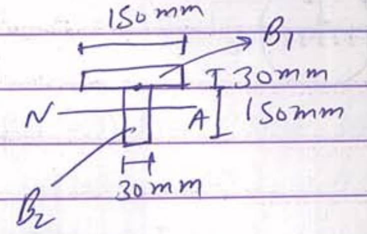
$I = \frac{\pi}{4}(c_1^4 - c_2^4) = \frac{\pi}{4}(50^4 - 20^4) = 4783074.82.$

$t = 60mm$

$\tau = \frac{4 \times 10^3 \times 155993.1585/2}{4783074.82 \times 60} =$ ~~2.174 MPa.~~ 1.09 MPa

(54) Problem 2





- Beam made of two boards.
- Determine max shear stress in glue necessary to hold boards together.
- Use $\bar{y} = 120$ mm
  $I = 27 \times 10^{-6}$ m^4.

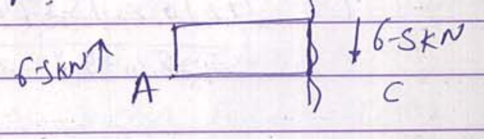
We need to calculate max shear first. $\left(\tau = \frac{VQ}{It}\right)$

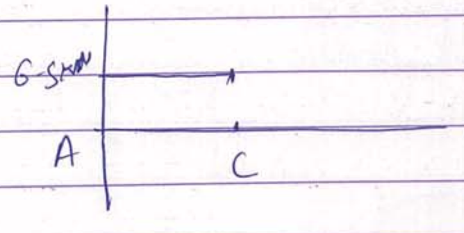
Here, $I = 27 \times 10^{-6} m^4$. $t = 30mm$
to find at glue $Q = A' \times \bar{y}'$

$$\bar{y}' = 15 + (150 - 120) = 15 + 30 = 45 \text{ mm}$$
$$Q = 150 \times 30 \times 45 \text{ mm}^3.$$

we only need to find V (max shear stress in beam)
(It has max shear stress, where there is max shear force)

$$R_A + R_B = 6.5 \times 4 = 26 \text{ kN}.$$
$$@M_A = 0 \quad -6.5 \times 4 \times 6 + R_B \times 8 = 0$$
$$R_B = \frac{6.5 \times 4 \times 6}{8} = 6.5 \times 3$$
$$R_B = 6.5 \times 3 \text{ kN}$$
$$R_A = 6.5 \text{ kN}.$$





From C to B

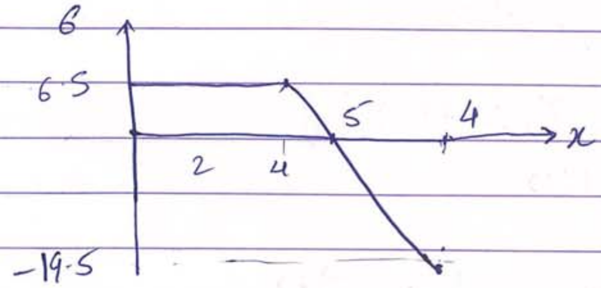
$$\frac{dV}{dx} = -w$$

$$= \frac{dV}{dx} = -6.5 \qquad V_2 - V_1 = -6.5x$$

$$V_2 = -6.5x + 6.5$$

@ $x = B$ $\quad V_2 = -6.5 \times 4 + 6.5 = -19.5\,kN.$

$V \uparrow \boxed{} \downarrow V_2$



So, $\boxed{} \uparrow 19.5\,kN.$

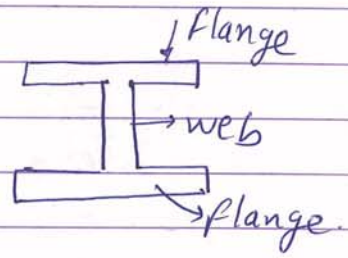
Max shear force $= -19.5\,kN.$, $V = 19.5\,kN.$

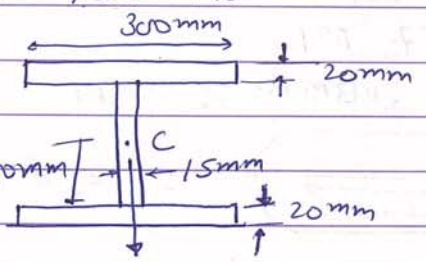
$$\tau = \frac{19.5 \times 10^3 \times 150 \times 30 \times 45}{27 \times 10^{-6} \times 10^{12} \times 30} = 4.875\,MPa.$$

(55) Problem 3

- Steel-I section is shown
- Draw the shear stress distribution across the cross-section for an applied shear force of 80 kN.

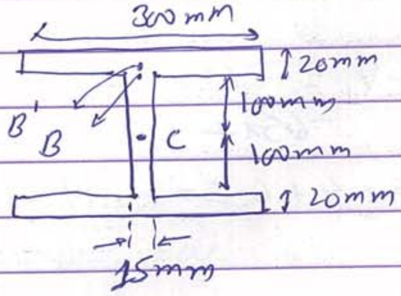
given $I = 155.6 \times 10^{-6}\,m^4$





3

$\tau = \frac{VQ}{It}$ τ @ top & bottom = 0.



$V = 80 kN$
$I = 155.6 \times 10^{-6} m^4$

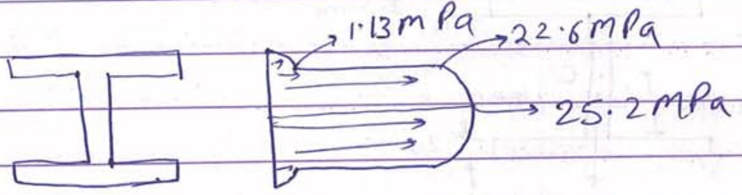
At B' $\tau = \frac{VQ}{It}$, $t = 300mm$

$Q = 300 \times 20 \times (100 + 10)$
$= 300 \times 20 \times 110$

$\tau_{B'} = \frac{80 \times 10^3 \times 300 \times 20 \times 110}{155.6 \times 10^{-6} \times 300 \times 10^{12}}$ MPa
$= 1.13$ MPa

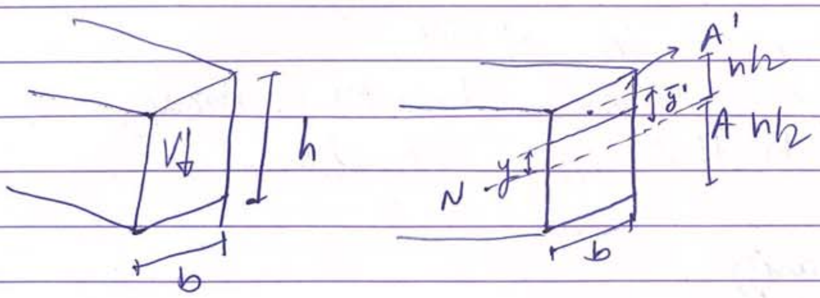
$\cancel{\tau_{B} = 80 \times 10^3 \times 100 \times 15 \times}$
for B, Q is same
$\tau_{B'} = \frac{80 \times 10^3 \times 300 \times 20 \times 110}{155.6 \times 10^{-6} \times 15 \times 10^{12}}$
$= 22.6$ MPa.

$\tau_C = \frac{80 \times 10^3 \times (300 \times 20 \times 110 + 100 \times 15 \times 50)}{155.6 \times 10^{-6} \times 15 \times 10^{12}}$
$= 22.6 \times 1.11$
$= 25.17$ MPa.



(56) Shear stress in rectangular beams

Most of the structure that we see are rectangular beams. So, we see variation of τ across height of beam.



we try to find τ at y. We know $\tau = \frac{VQ}{It}$

$$Q = A' \cdot \bar{y}'$$

$$A' = b \times \left(h/2 - y \right) ; \quad \bar{y}' = \frac{1}{2}\left(\frac{h}{2} - y\right) + y$$
$$= \frac{1}{2}\left(h/2 + y\right)$$

$$Q = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$I = \frac{bh^3}{12} \quad ; t = b$$

$$\tau = \frac{V \times \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)}{\frac{bh^3}{12} \times b} = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$

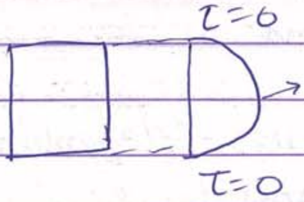
We have quadratic variation. At $y = 0$, $\boxed{\tau = 1.5 \frac{V}{A}}$

$V/A = \tau_{avg}$ so, at $y=0$ i.e at NA $\boxed{\tau = 1.5\,\tau_{avg}}$.

So, at neutral axis, shear stress is 1.5 times the average shear stress.

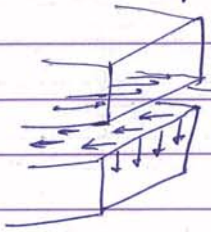
At $y = h/2$, $\tau = 0$. & at $y = -h/2$ $\tau = 0$.

At top of beam, no force is acting in longitudinal direction, so the complimentary shear has to be 0. And that is what we get. i.e. at top & bottom $\tau = 0$.



$\tau_{max} = 1.5\ V/A$ at NA.
Size of arrows at center is more than that of at the edges.

Due to complementarity



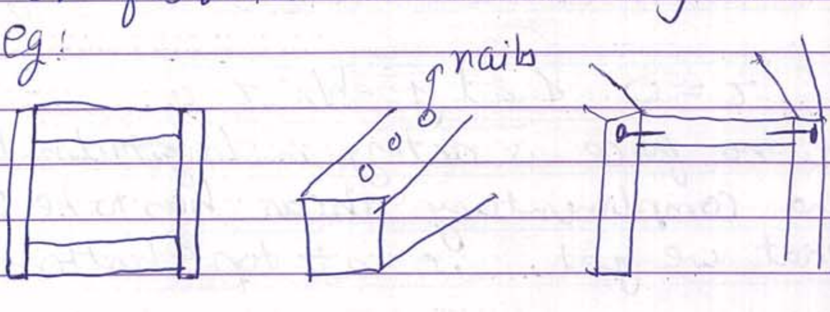
In old houses, if it has wooden beam, we can observe crack at center of wood.

So, for rectangular beam -
- max shear is much higher (1.5 times) than the average shear.
- Zero shear at top and bottom (because of no tangential shear loads)

(57) Shear Stresses in Built up Sections

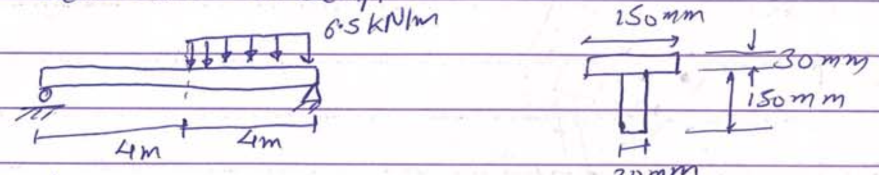
Built up members - Structures "built up" of multiple parts that are glued or nailed or fastened or welded together.

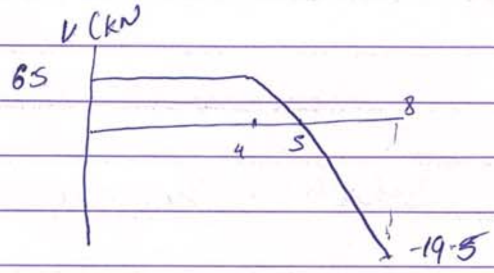
eg:



Nails, glues, fasteners etc. required to resist shear and prevent members from sliding against one another.

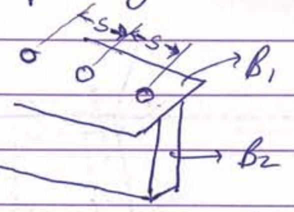
Example →
We saw an example



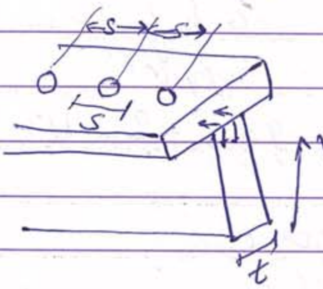


$V = 19.5\,kN$
So, we calculated shear
The plane contains glue & glue resists shear.

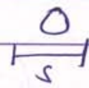
Instead of glue if we put nails. The job of nails is the same, i.e. to resist sheer. We need to find spacing b/w nails.



We use shear flow to calculate fastener/nail spacing.



Due to V, there is also longitudinal shear. Each bolt resist shear ~~at~~ to a distance s.



Total force resisted by bolt $= \underline{\tau \times s \times t}$.

$\tau \times t \times s = F$.

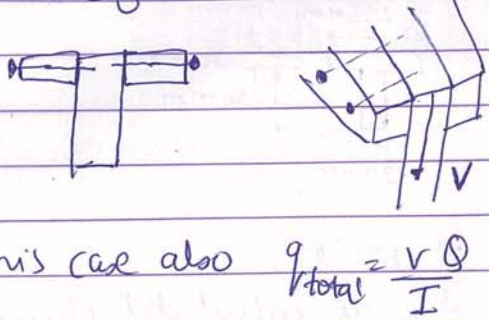
$F = q \times s \qquad q = \tau \times t$

q is $\qquad \tau = \dfrac{VQ}{It}$; $\boxed{q_{total} = \dfrac{VQ}{I}}$.

$$\boxed{q_{total} = \frac{VQ}{I} = \text{shear force per unit length}}$$

↳ shear flow.

Now, instead of one row of bolts what if we have two rows of bolts.



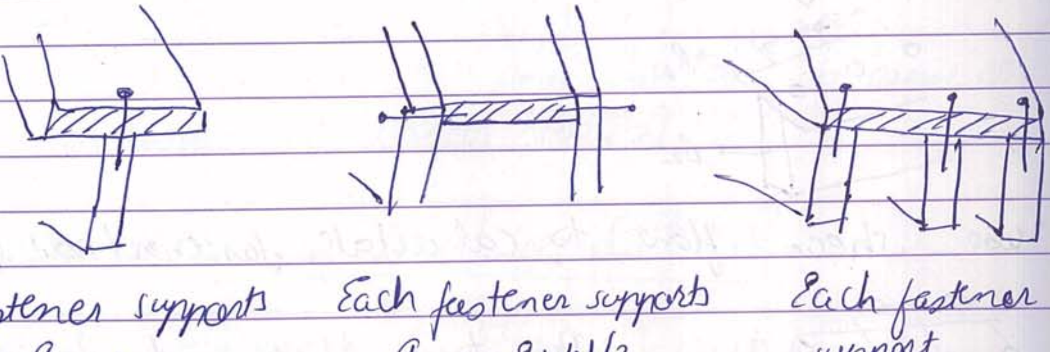
In this case also $q_{total} = \frac{VQ}{I}$

Each row of bolts resist q_{each}, which in this case equal $q_{total}/2$.

For each row of bolt $\frac{q_{total}}{2} s = F$

$q_{each} = q_{total}/2$.

So,



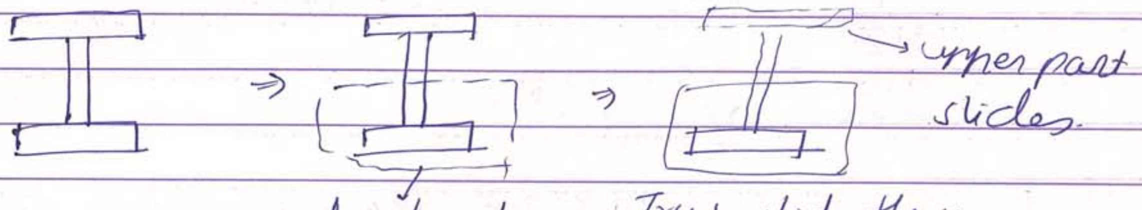
Each fastener supports $q = q_{total}$ | Each fastener supports $q_{each} = q_{total}/2$ | Each fastener support $q_{each} = q_{total}/3$.

In formulation, V is entire shear force, I is for entire surface

How do we get Q?

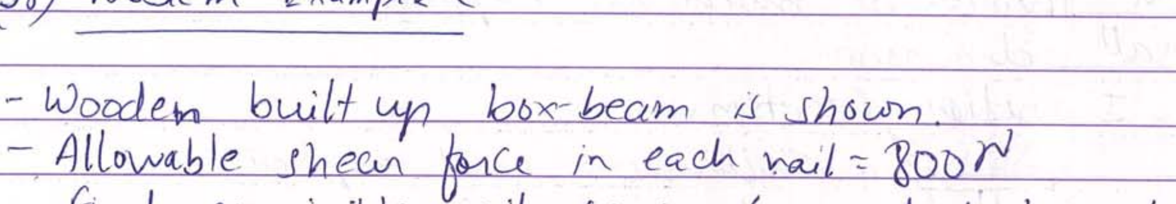
Each nail is resisting the part that is being slided off. (So, the shaded portion is for which we get Q)

- Which area to consider for Q?
  - Consider the 'loose' parts of the built-up section that are likely to be sheared off.
  - Calculate Q for the areas of these "loose" parts



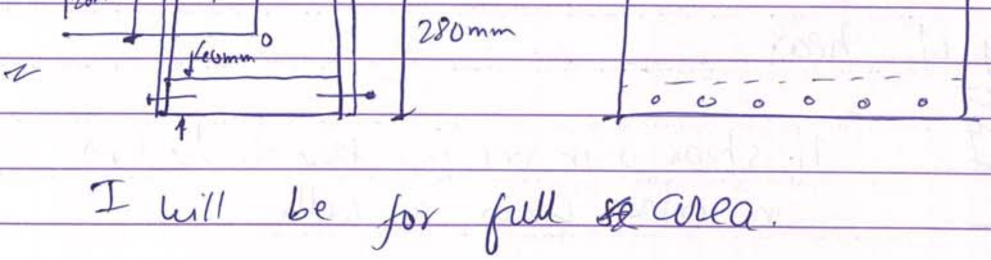
Process →

- Locate NA (Neutral Axis)
- Fix structure below NA
- Apply a push to the cross-section
- See which part is about to slide (shear) off
- Calculate Q for that area



(58) Problem Example C

- Wooden built up box-beam is shown.
- Allowable shear force in each nail = 800N
- Find permissible nail spacing for applied shear force $V = 10.5$ KN.



I will be for full ~~se~~ area.

$$q_{total} = \frac{VQ}{I}$$

$$Q = A'y' = 180 \times 40 \times 120 = 864000 \ mm^3$$

$$I = \frac{1}{12} \times 240 \times (284)^3 - \frac{1}{12} \times 180 \times (200)^3$$

$$= 264.2 \times 10^6 \ mm^4$$

$$q_{total} = \frac{VQ}{I} = \frac{10.5 \times 10^3 \times 864000}{264.2 \times 10^6} = 34.33 \ MPa$$

$$q_{each} = \frac{q_{total}}{2} = 17.17 \ MPa$$

Allowable shear force = 800 N

$$q_{each} \times s = F$$

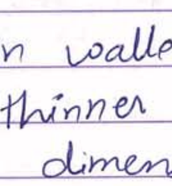
$$17.17 \times s = 800$$

$$s = \frac{800}{17.17} = 46.59 \ mm$$

(59) <u>Shear flow in thin walled sections (Part 1)</u>

- For thin walled sections, thickness of members are much thinner or much less compared to overall dimension.

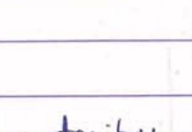
eg :- I section, C section.



→ Thicknes are small compared to other dimension.

<u>Recap</u>

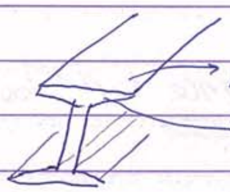
- Complimentarity of shear



If shear is in one face then it develops in other faces as well.

For rectangular beam ~~the~~ shear stress at top and bottom are 0. and maximum at the center near neutral axis.

For I section, as there is no shear on top face so due to complimentarity, there will be no shear of ~~it~~ on the face $\perp^{r}$ to it

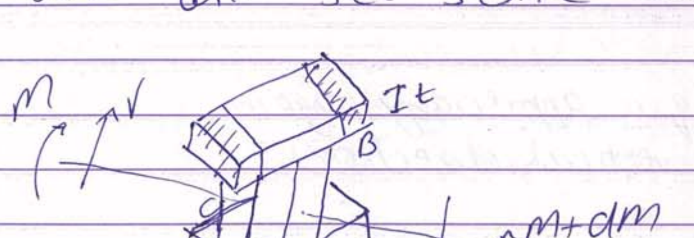


Similarly, 

But there will be shear in vertical direction.

Shear flow = $q = \tau \times t = \frac{VQ}{I}$

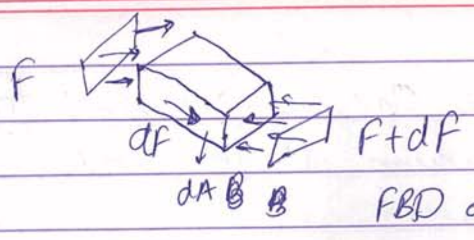
We will see some thin walled sections.

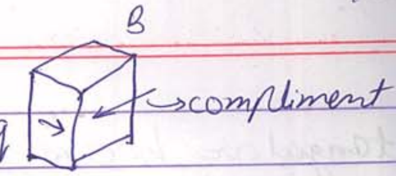


Due to M & M+dM, the, we are going to have Normal stresses as ($\sigma = My/I$).
So, this normal stresses will be highest at top & bottom.

So, if we have different moment, what in the section takes that difference.

For section in member →

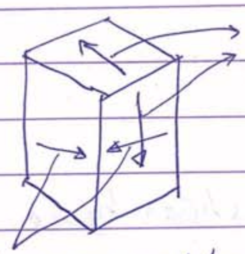




FBD at B (B is small section in I-section)

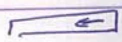
As we have shear, we due to complimentarity we will have on the other face as well.

So, for flange element



q is negligible since top & bottom are stress free

q is appreciably large sideways.

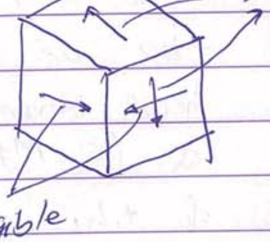
So, in flange shear acts parallel to surface 

In the web, 

, on face A there is no shear.

So, due to complementarity → dir^n on B face shear will be 0.

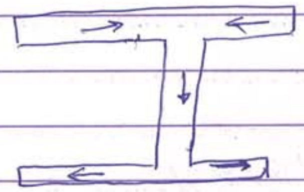
Web element



q is appreciably large in vertical direction.

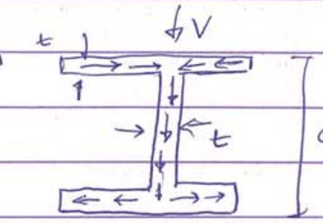
q is negligible since side faces are stress free

So, in web, it acts down & again $\parallel^{el}$ to surface of web



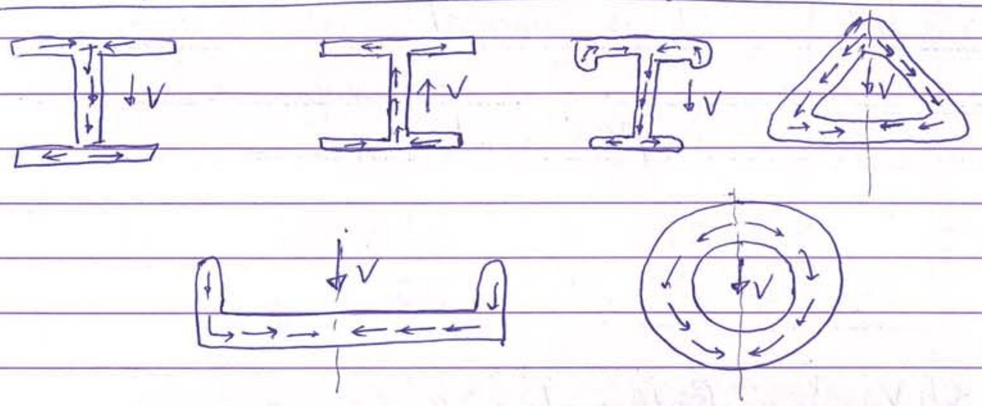
∴ Shear flow q acts parallel to element sides.

This looks as →



It is like flow of water

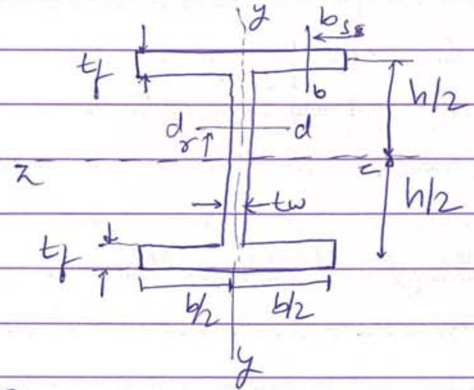
Flow of shear in thin walled single unit members



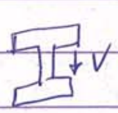
Shear flow q acting $\parallel^{el}$ to element sides.

(60) Shear flow and Shear Stress in I-sections (Part 2)

Let us take the I-section.

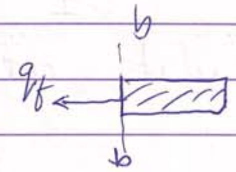


As thickness is small, dimensions are marked center to center.



Flange

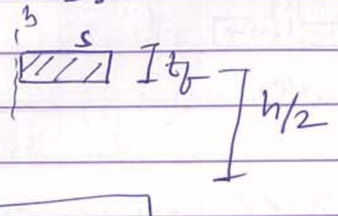
we take section b-b.



$q_f = \frac{VQ}{I}$

$I \rightarrow$ property of I-section (not changing)
$V \rightarrow$ applied shear force

So, we need Q.



$A' \times y'$
$Q = s \times t_f \times h/2$

$$\boxed{q_f = \frac{s t_f h V}{2I}}$$

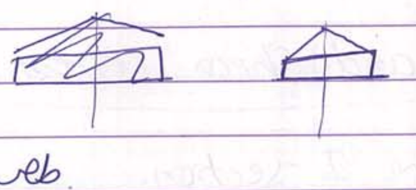
It is symmetric about y-y axis. If $s = 0$, $q_f = 0$. it peaks at $s = b/2$

$q_f = \tau \times t_f \Rightarrow \tau_f = \frac{q_f}{t_f} = \frac{shV}{2I}$

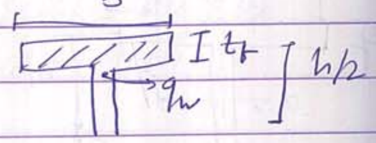
$$\boxed{\tau_f = \frac{shV}{2I}}$$

Both values q_f & τ_f are 0 at the edge.

So, for flange



Now, we look for web



@ interface with flange

$q_w = \frac{VQ}{I}$

For, Q, we calculate area times y above point

$Q = b \times t_f \times h/2$

So, $\boxed{q_w = \frac{b t_f h V}{2I} \Rightarrow \tau_w = \frac{b t_f h V}{t_w \times 2I}}$

Now, if we calculate at center, we get,

$\tau_w = \left(\frac{b h t_f}{t_w} + \frac{h^2}{4} - r^2\right)\frac{V}{2I}$ (r is distance from center)

$q_w @ b/2 = \frac{bht_f V}{2I} \leq q_f @ b/2 = \frac{bht_f V}{4I}$; ~~$q_w @ b/2 = bht_f$~~

if $r = h/2$, we get τ_w @ flange-web interface.

So, In Flange,

$$\tau_f = \frac{shV}{2I} \quad \& \quad q_f = \frac{sht_f V}{2I}$$

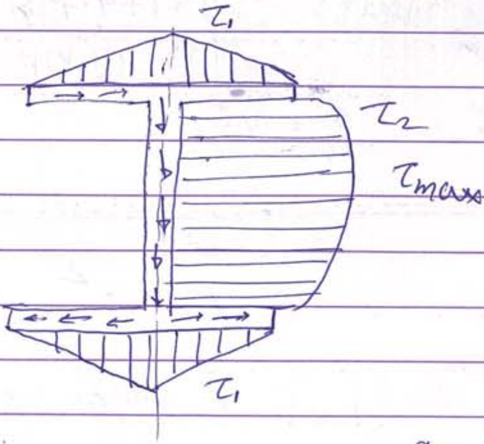
(zeros at edge, max at center)

In Web,

$$\tau_w = \frac{bht_f V}{2I t_w} \quad \& \quad q_w = \frac{bht_f V}{2I} \text{ (at flange-web intersection)}$$

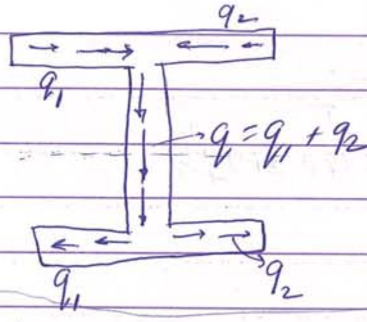
$$\tau_w = \cancel{bht_f}\left(\frac{bht_f}{t_w} + \frac{h^2}{4} - r^2\right)\frac{V}{2I}$$

(max at middle of web)



This represents only magnitude & not dir^n.

Also note:

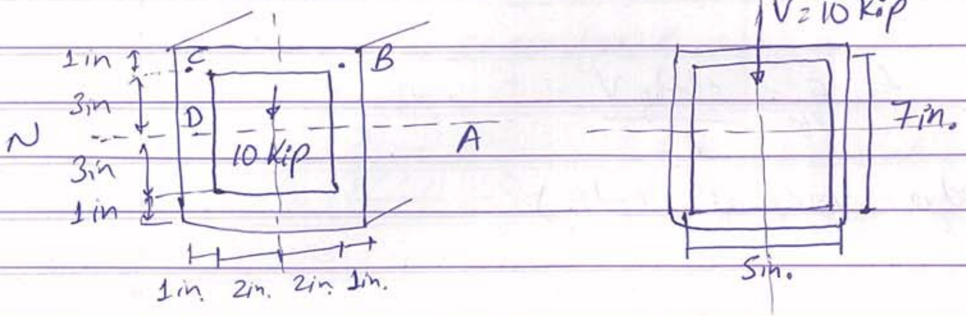


Just like flow of water

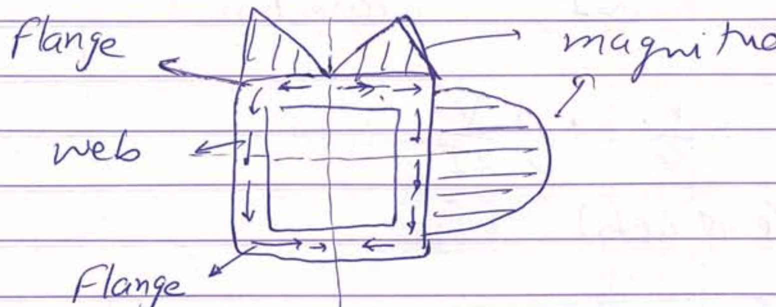
NB! → Shear flow has linear variations in segment perpendicular to V & parabolic variations in segments parallel to V

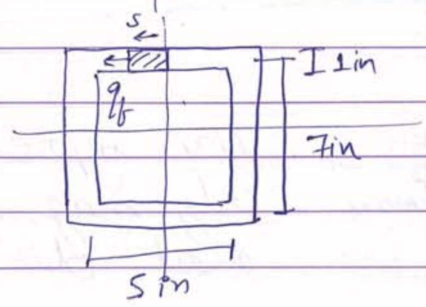
(61) Problem Example

Determine the variation of shear flow through the cross-section.



We expect shear flow to be like





$I = 179.7\ in^4$

$V = 10$ kip

$$q_f = \frac{VQ}{I}$$

$$Q = \underbrace{(s \times 1) \times \frac{7}{2}}_{A' \times y'} = 3.5s$$

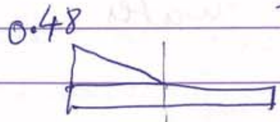
$q_f = \frac{35s}{I}$, so it is a linear variation.

@ $s = 2.5$ $\quad q_f = \frac{35 \times 2.5}{I}$

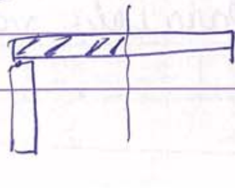
$= 0.48$ kip/in ~~lb/in~~

$q_f = 0$ @ $s = 0$

$q_f = max$ @ $s = 2.5$



Now, let us see for web. For web total area to consider =



Web q_w @ interface $= \frac{VQ}{I}$

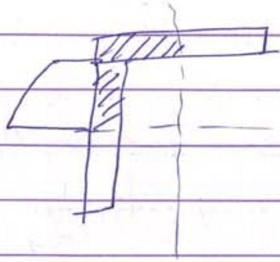
$= \frac{10 \times Q}{179.7}$ $\quad Q = 2.5 \times 1 \times 3.5$

$=$ 0.48 ~~lb/in~~ kip/in

So, it is like flow of water. Whatever flows goes to other side.

Let us see for middle of web.

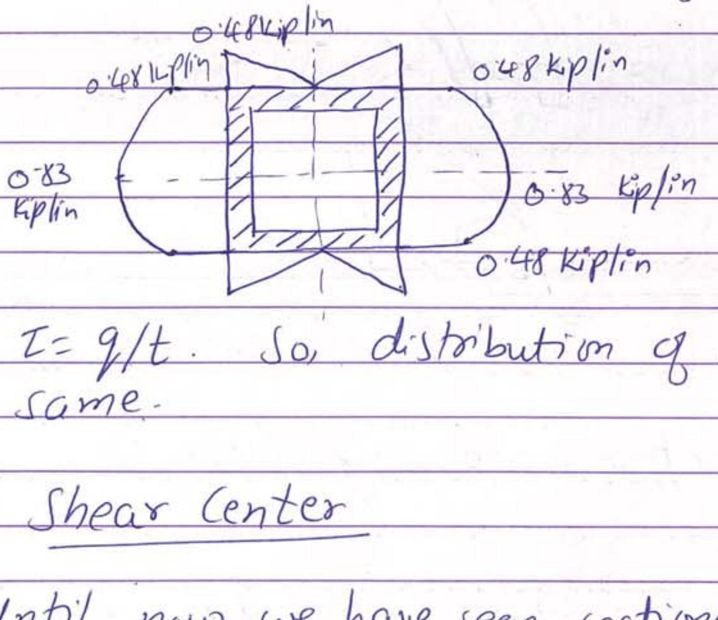
q_w @ D $= \frac{VQ}{I} = \frac{10 \times Q}{179.7}$



$Q = 2.5 \times 1 \times 3.5 + 3.5 \times 1 \times 3.5$
$= 8.75$

q_w @ D $= \frac{10 \times 8.75}{179.7} = 0.83$ kip/in

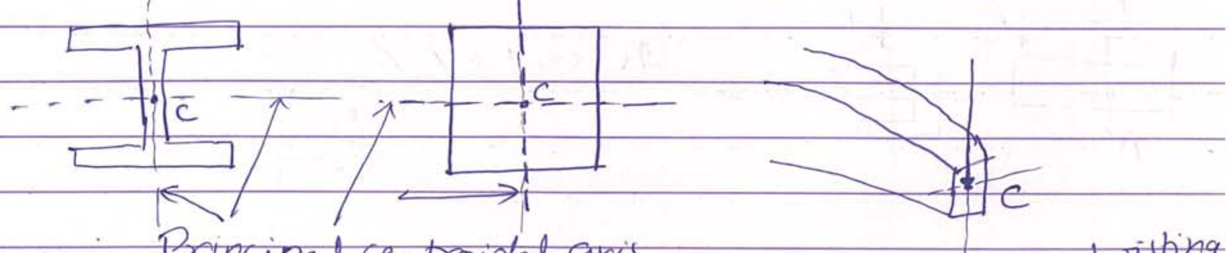
Variation is parabolic. Shear flow overall looks as →



$\tau = q/t$. So, distribution of shear stress is also same.

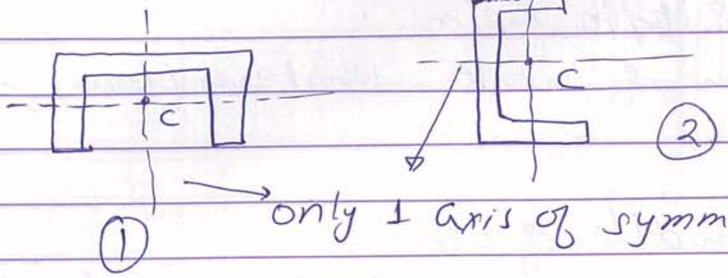
(62) Shear Center

- Until now we have seen sections that have principal centroidal axis coincide with axis of symmetry.

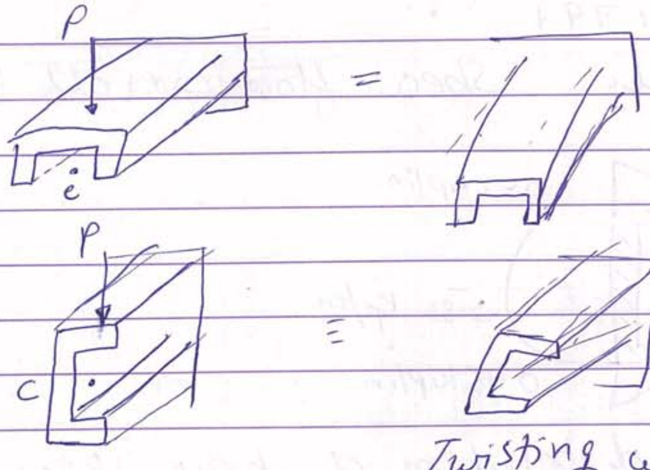


This type of beam do not under go twisting due to shear force.

But consider a C-channel.



Now, if we apply load passing through centroid axis of symmetry, ~~ther~~ there is only bending, no twisting for case ①, but it is not for case ②.

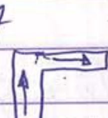


Twisting with bending

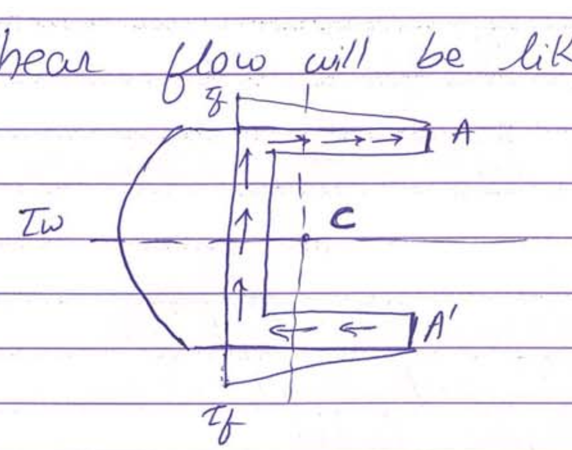
NB! Thin walled sections are very weak in torsion.

• Consider a channel (C) section with load P acting and 1 end as fixed

At section A-A

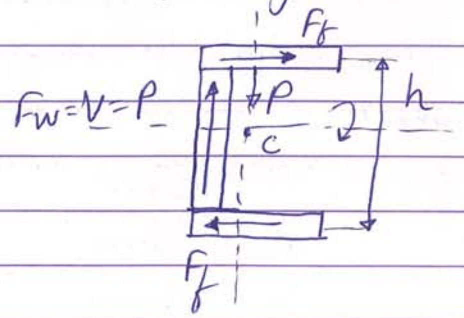


Shear flow will be like



At A & A' edges shear is 0.
Shear stress at other portion will be q/t.

The force ~~d~~ diagram looks as →

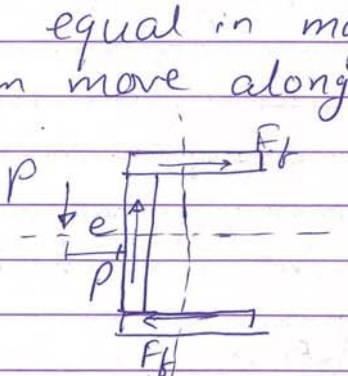


$F_f = \frac{\tau_f \times t \times \text{length}}{2}$

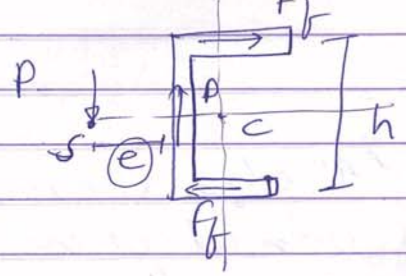
(Area under curve)

The two F_f are forces separated by a distance h. Similarly P & P(V=P) are separated by same distance. These two forces form a couple. So, as a result of couple you are getting that amount of twisting.

Now, how do we resist twisting. In our hand we have the location of P.
If $F_f \times h$ is the internal moment that causes torsion or twisting
To cancel this, we must have an anticlockwise moment equal in magnitude to $F_f \times h$.
(We can move along any side of beam)



Can there be a location where P can be applied and there will be no twisting? True.

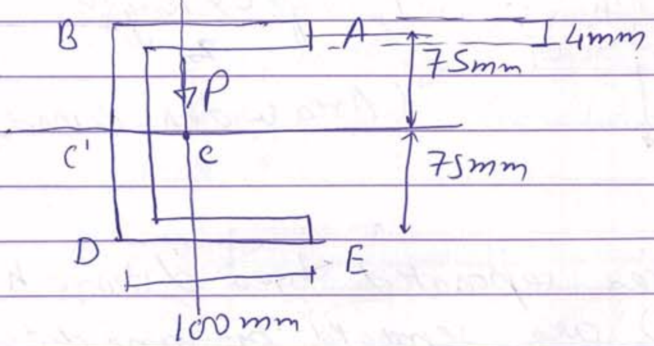


$P \times e = f_f \times h$

e is distance of shear center

Shear center is the distance at which you should apply the load P such that beam only bends and does not twist.

(63) Problem Example

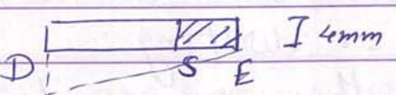


$P = 10 kN$.
$h = 150 mm$
$b = 100 mm$.
$I = 5625000 mm^4$.

We need to find the F_f first. So, we need shear stress. It will be linear as Flange is $\perp^{er}$ to applied force.

$$q = \frac{VQ}{I} \qquad V = 10 kN,\ I = 5625000 mm^4$$

We need Q.



$$Q = 75 \times s \times 4 \quad (A' \times y')$$
$$= 300 s$$

$$q = \frac{10 \times 10^3 \times 300 s}{5625000} = \frac{3000 s}{5625} = 0.5335 s$$

@ E $q_f = 0$ @ D $q_f = 53.35$ N/mm

$\tau_f = 53.35 / 4$ N/mm²

@ web q_w @ D = q_f @ D = 53.35 N/mm

$\tau_f = 53.35/4$ MPa

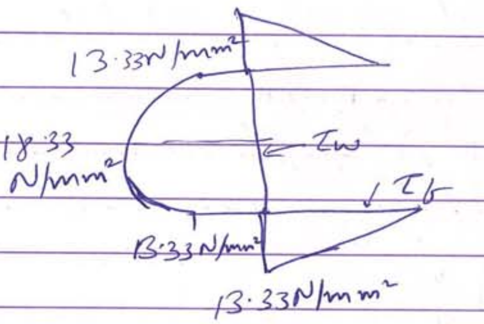
@ C' $q_w = \frac{VQ}{I}$

$Q = (100 \times 4 \times 75 + 75 \times 4 \times 75/2)$

@ c' $q_w = 73.33$ N/mm

So, $\tau_w = \frac{73.33}{4} = 18.33$ N/mm

Our shear stress distribution looks as



So, $F_f = \frac{\tau_f \times t \times b}{2}$

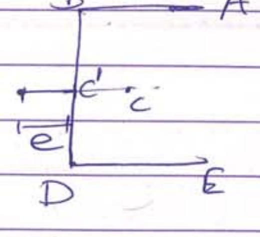
$= \frac{1}{2} \times \frac{53.35}{\cancel{4}} \times \cancel{4} \times 100$

$= 2667.5$ N.

$P \times e = F_f \times h$

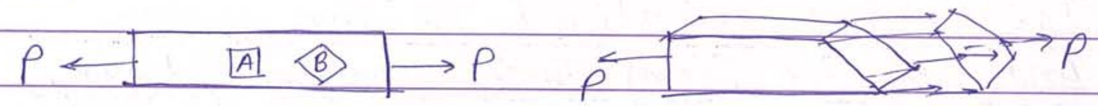
$e = \frac{2667.5 \times 15\cancel{0}}{1000\cancel{0}} = 40$ mm.

So, location of shear center is 40mm from C'.



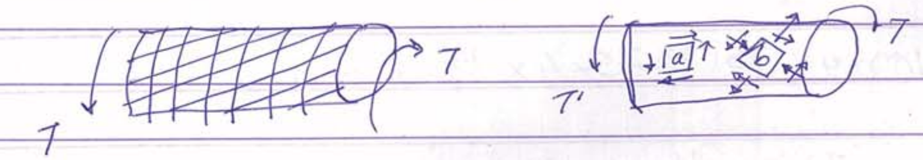
(64) Basic Concepts of Stress Transformation

Recap



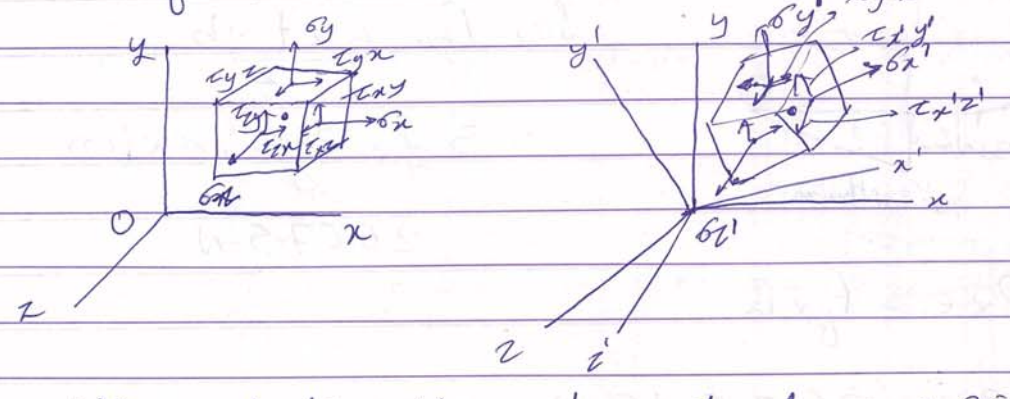
For wood we saw it is weak in shear. Shear tends to peak up when θ was 45°. So, at for the same point at different angle we get different stress.

We also saw case for torsion



In chalk breaking we get it at 45°.

Main objective here is → how do stresses (and strains) in an element get transformed about a rotated set of axis

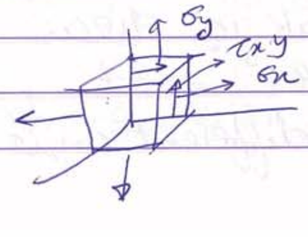


We rotate the element at same point or we take a different plane of section, how do the stresses & strain get transformed.

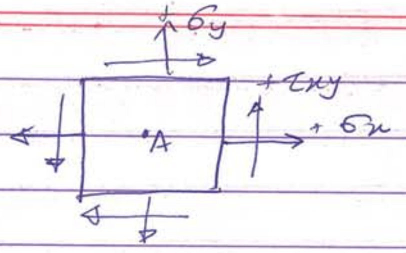
Why is it important?

Because maximum forces, and failures may happen along ~~different~~ directions different from the applied force and chosen coordinate system

We will first see for 2D. For plain stress $\sigma_z = 0$, $\tau_{xz} = 0$ & $\tau_{yz} = 0$. This is relevant for Thin members. And even for beams under transverse loading, the stresses in z-dir^n ($\perp^{er}$ to applied load) tend to be 0.



2D element



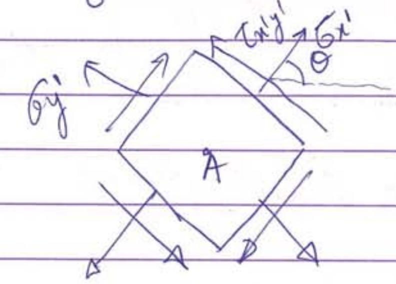
(All shear stress are +ve & all normal stresses are +ve)

Sign convention →

σ_x, σ_y → +ve Tensile ; θ → +ve clockwise

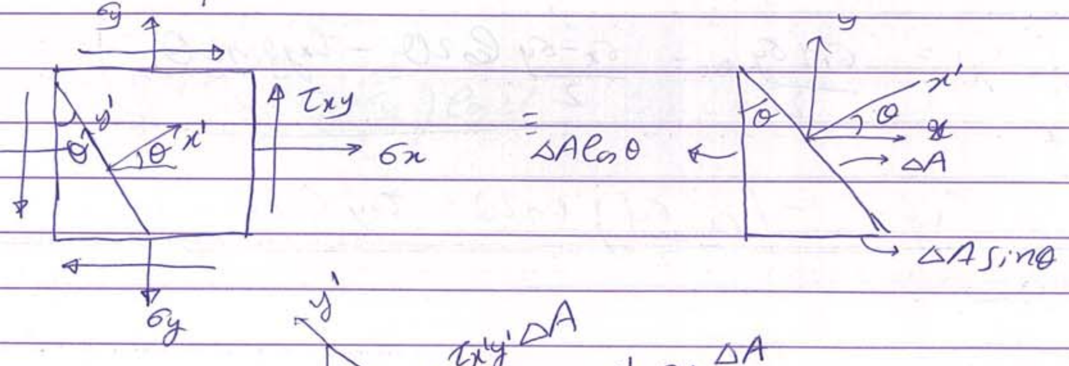
τ_{xy} { +ve when generating on a +ve face & acting along +ve dir^n
→ +ve when generating on a -ve face & acting along -ve dir^n
→ -ve otherwise

Now, if we rotate about θ. (θ is +ve for anticlockwise)

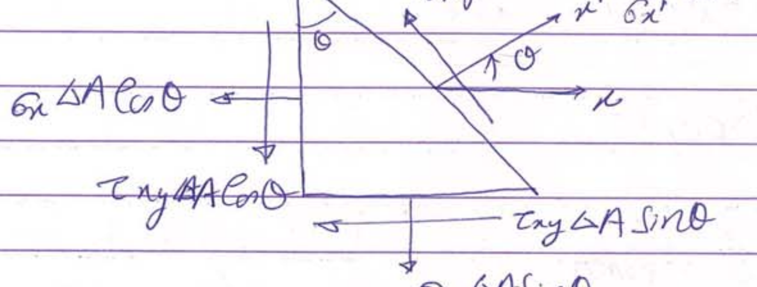


This is for the same point within the body.

||| equivalent



FBD:



$\Sigma F_{x'} = 0$

$\sigma_{x'}\Delta A - (\tau_{xy}\Delta A \sin\theta)\cos\theta - (\sigma_y \Delta A \sin\theta)\sin\theta$
$\quad - (\tau_{xy}\Delta A\cos\theta)\sin\theta - (\sigma_x \Delta A \cos\theta)\cos\theta = 0$

$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta}$$

$$\boxed{\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta}$$

For the same diagram $\Sigma F_{y'} = 0$

$$\tau_{x'y'}\Delta A + (\tau_{xy}\Delta A\sin\theta)\sin\theta - (\sigma_y\Delta A\sin\theta)\cos\theta - (\tau_{xy}\Delta A\cos\theta)\cos\theta + (\sigma_x\Delta A\cos\theta)\sin\theta = 0$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\boxed{\tau_{x'y'} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta}$$

For $\sigma_{y'}$, in eq^n for $\sigma_{x'}$ replace θ with $90+\theta$.

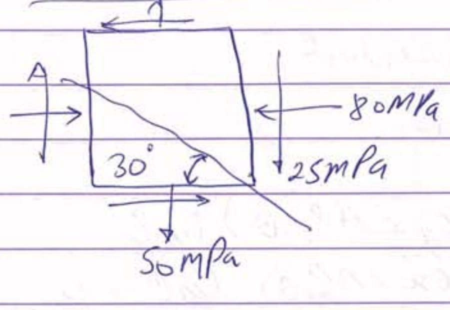
$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{xy} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

NB! $\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$

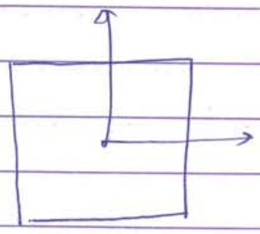
(65) Problem Example

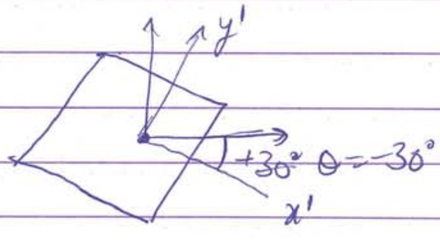


- State of stress for an element on the body is shown
- Represent state of stress of an element at same location oriented at 30° clockwise

The element is rotated clockwise 30°.

$\sigma_x = -80\ MPa$ (Compressive)
$\sigma_y = +50\ MPa$
$\tau_{xy} = -25\ MPa$ (+ve face, −ve dirⁿ)





$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \left(\frac{-80+50}{2}\right) + \left(\frac{-80-50}{2}\right)\cos(-60^\circ) + (-25)\sin(-60^\circ)$$

$$= -15 - 65 \times \frac{1}{2} + 25 \times \frac{\sqrt{3}}{2} = -15 - 32.5 + 12.5\sqrt{3}$$
$$= -25.8\ MPa.$$

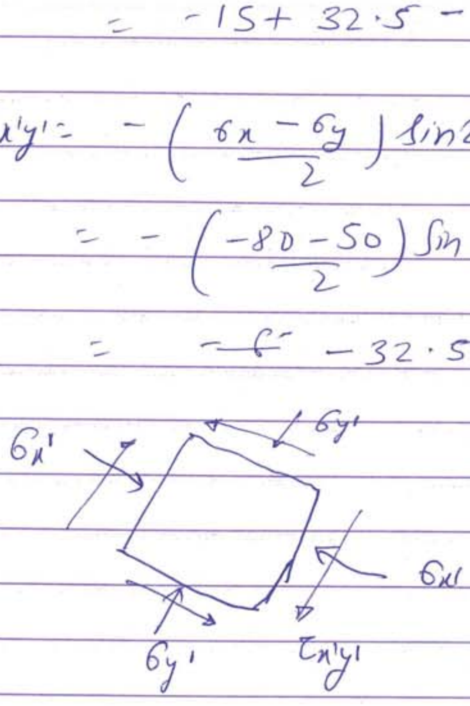
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= -15 - (-65)/2 - (-25)\sin(-60)$$
$$= -15 + 32.5 - 12.5\sqrt{3} = -4.15\ MPa.$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\left(\frac{-80-50}{2}\right)\sin(-60) + (-25)\cos(-60)$$

$$= \cancel{-6} - 32.5 - 12.5\sqrt{3} = -68.8\ MPa.$$



(66) Principal Stresses & Max In-plane shear

After rotation, there must be some angle $\theta = \theta_p$ at which the stresses $\sigma_{x'}$ & $\sigma_{y'}$ reach their maximum & minimum values respectively.
One reaches max so other reaches minimum,
As we know

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

So, if $\sigma_{x'}$ is maximum then $\sigma_{y'}$ is minimum
we find $\partial x'/\partial\theta$, $\partial y'/\partial\theta$

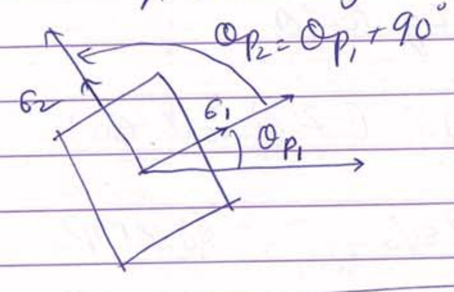
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_{x'}}{d\theta} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta = 0$$

$$\boxed{\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}}$$

θ_p; p represents principal direction.

Two angles $2\theta_p$ separated by $180°$. P refers to 'principal' where max stress (σ_1) & min stress (σ_2) occurs.
θ_p is separated by $90°$.



After substituting value of θ_p, we get

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\boxed{\sigma_1 + \sigma_2 = \sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

The planes where these maximum stresses act are called principal planes.

When an element is rotated ~~by~~ about the principal direction, the ~~str~~ shear stress vanish.

When we have σ_1 & σ_2, $\tau_{xy}=0$. We only have maxima & minima:

$$\tau_{x'y'} = 0 \text{ at } \theta_{p_1} \text{ \& } \theta_{p_2}.$$

Summary:

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} \rightarrow \text{Gives principal stresses } (\sigma_1 \text{ \& } \sigma_2)$$

$$\tan 2\theta_p = \tau_{xy} / (\sigma_x - \sigma_y)/2 \rightarrow \text{Gives principal planes } (\theta_{p_1} \text{ \& } \theta_{p_2})$$

$$\tau_{x'y'} = 0 \text{ on principal planes} \rightarrow \text{No shear at } \theta_{p_1} \text{ \& } \theta_{p_2}.$$

Now, there must be some angle θ_s at which the stresses $\tau_{x'y'}$ reaches the maximum value.

$$\frac{d\tau_{x'y'}}{d\theta} = -\left(\frac{\sigma_x-\sigma_y}{2}\right) 2\cos 2\theta - 2\tau_{xy} \sin 2\theta = 0$$

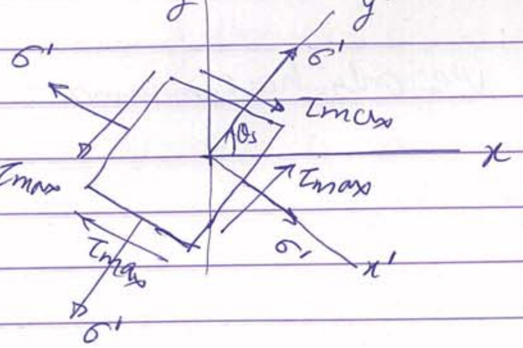
$$\boxed{\tan 2\theta_s = -\frac{(\sigma_x-\sigma_y)/2}{\tau_{xy}}}$$

⇒ When we compare with principal stress dir^n i.e. θ_p we see that θ_s & θ_p are separated by 45°

∴ Plane of maximum shear is located at 45° from the principal plane.
But the σ in plane of maximum shear do not vanish.

At max shear planes:

$$\boxed{\tau_{max} \; (\tau_{x'y'max}) = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \quad \& \quad \boxed{\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}}$$

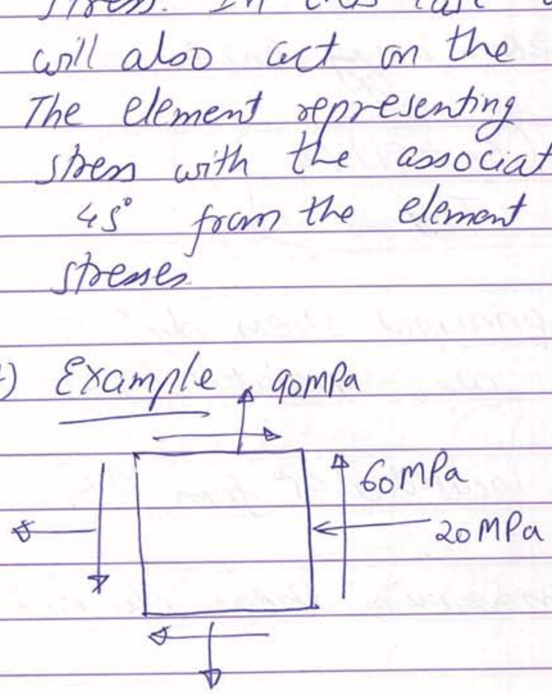


∴ At planes of maximum shear, normal stresses still exist.

Important Points:

- The principal stresses represent the maximum & minimum normal stress at the point.
- When the state of stress is represented by the principal stresses, no shear will act on the element.
- The state of stress at the point can also be represented in terms of maximum in-plane shear stress. In this case an average normal stress will also act on the element.
- The element representing the maximum in-plane shear stress with the associated normal stress is oriented 45° from the element representing the principal stresses.

(67) Example



⇐ State of stress at failure point
- Represent state of stress in terms of principal stress. Show diagram.

- Represent state of stress in terms of max in-plane shear and associated average normal stress.

$\Rightarrow$ $\sigma_x = -20\,MPa$ $\sigma_y = 90\,MPa$ $\tau_{xy} = +60\,MPa$

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + (\tau_{xy}^2)}$$

$$= \frac{90-20}{2} \pm \sqrt{\left(\frac{-110}{2}\right)^2 + 60^2}$$

$$= 35 \pm \sqrt{55^2+60^2} = 35 \pm 81.39.$$

$$\sigma_1 = 116.4\ MPa \qquad \sigma_2 = -46.4\ MPa.$$

For angle $\tan 2\theta_p = \dfrac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)} = \dfrac{60}{\frac{-20-90}{2}} = \dfrac{-120}{110}$

$= -1.09$

One root $\theta_p = -23.73°$

other root $\theta_p = 90+(-23.73) = 66.27°$

The two angles are separated by $90°$.

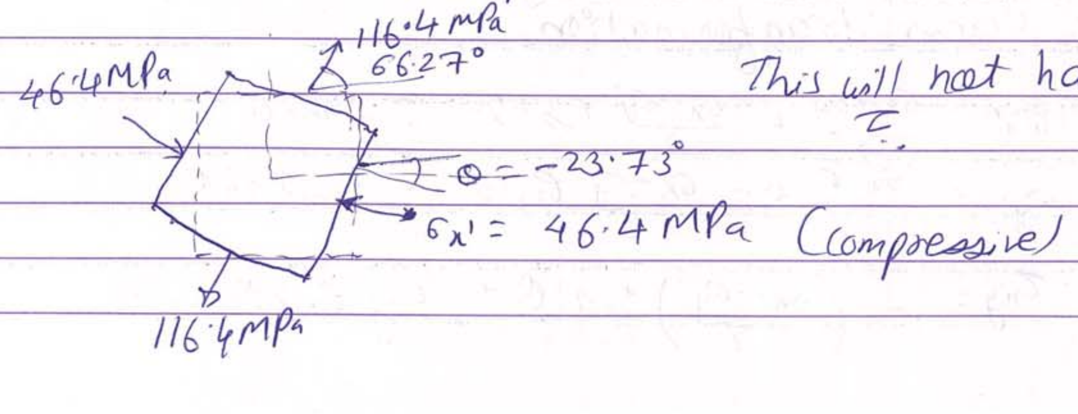
Let us put $-23.73°$ into basic eq^n

$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{-20+90}{2} + \left(\frac{-20-90}{2}\right)\cos(2\times -23.73) + 60\times\sin(2\times -23.73)$$

$$= -46.4\ MPa.$$

Now, we know which angle corresponds to which stress. 116.4 corresponds to 66.27°



This will not have τ.

For max in plane shear, it will be 45° away from the principal plane.
Let us see if we get the same.

$\sigma_x = -20\,MPa \quad \sigma_y = 90\,MPa \quad \tau_{xy} = 60\,MPa$

$\therefore \dfrac{\sigma_x+\sigma_y}{2} = 35\,MPa \quad \therefore \dfrac{\sigma_x-\sigma_y}{2} = -55\,MPa.$

$$\tau_{x'y'\,max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{(-55)^2 + 60^2}$$

$$= 81.39\,MPa$$

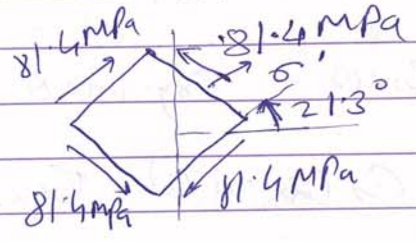
$\sigma' = 35\,MPa.$

$$\tan 2\theta_s = -\left(\frac{\sigma_x-\sigma_y}{2}\right) / \tau_{xy}$$

$$\tan 2\theta_s = -(-55)/60 = 55/60$$

$\Rightarrow \theta_s = 21.3°.$

$\theta_{s_2} = 21.3 + 90 = 111.3°$



$21.3° + (23.73°)$
$= 45°.$

(68) Mohr's Circle Concept (Part 1)

Mohr's circle $\Rightarrow$ A graphical tool for Stress transformation.

$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \quad -(2)$$

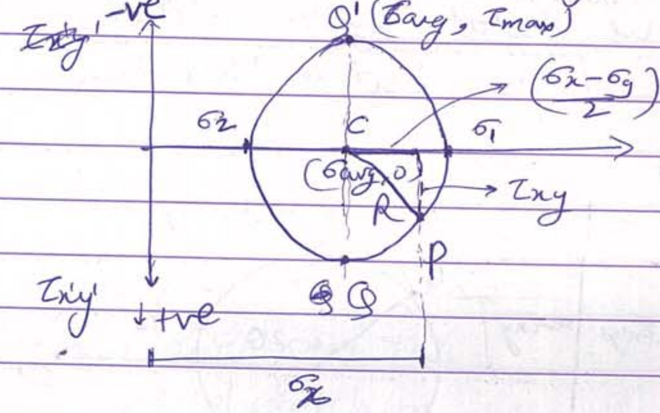
$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad -(1)$$

If we squared & add (1) & (2), we get

$$\Rightarrow \boxed{\left(\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Let $(\sigma_x + \sigma_y)/2 = \sigma_{avg}$ $\quad R^2 = ((\sigma_x - \sigma_y)/2)^2 + \tau_{xy}^2$

we get $(\sigma_{x'} - \sigma_{avg})^2 + \tau_{x'y'}^2 = R^2$ → eq^n of circle.
centered at $(\sigma_{avg}, 0)$



$R = \sqrt{((\sigma_x - \sigma_y)/2)^2 + \tau_{xy}^2}$

σ_1, σ_2 are principal stresse.

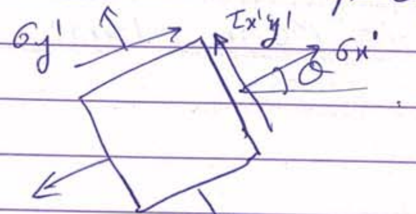
Q is the point of max shear

Q' is the point of minimum shear

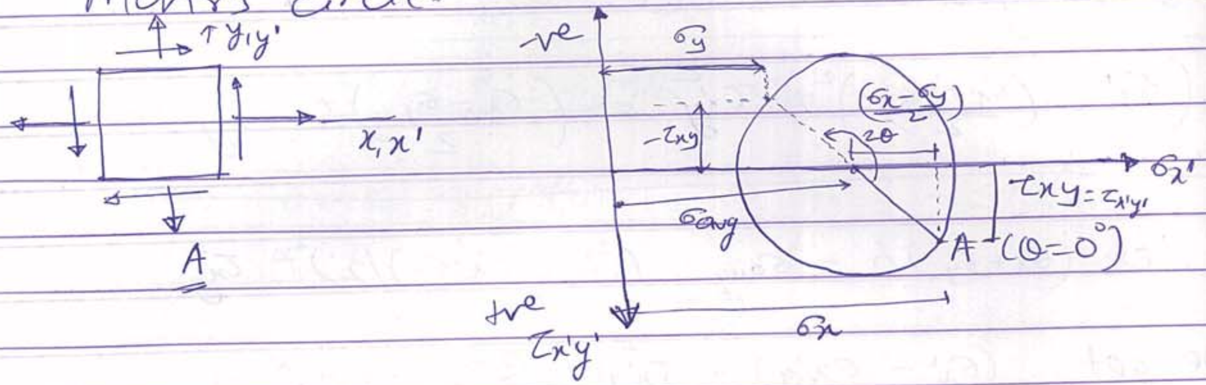
σ is +ve to right. (Different book follow
τ is +ve downwards. different conventions)

(69) Part 2

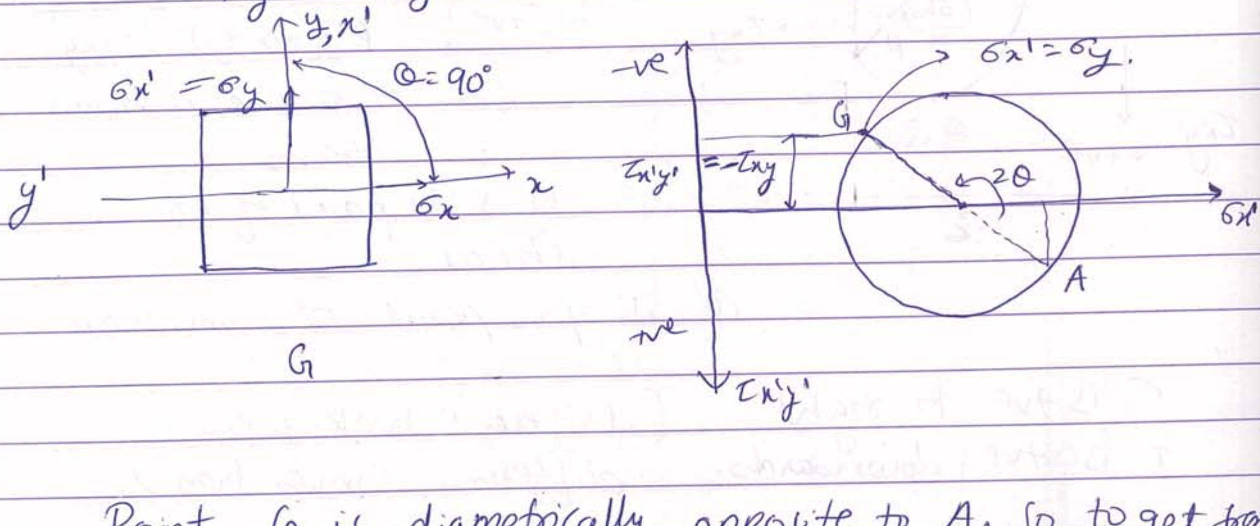
Each point of the Mohr's circle represents two stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on the side of the element side defined by x' axis, when the axis is in a specific dir^n θ.



x' axis can be anywhere, ~~& it can also~~ as we rotate we move to different points on Mohr's circle.



~~In mohr~~ If we rotate element by 90°, we get σ_y'. So, when we rotate by 90° in Mohr circle we rotate by 180°. But in this case $\tau_{x'y'}=-\tau_{xy}$



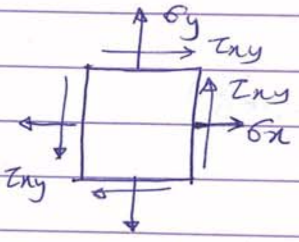
Point G is diametrically opposite to A. So, to get from 0 to 90° in element, we have to go 2θ in Mohr circle.

① So, a rotation of θ in original elements, leads to rotation of 2θ in Mohr's plane.

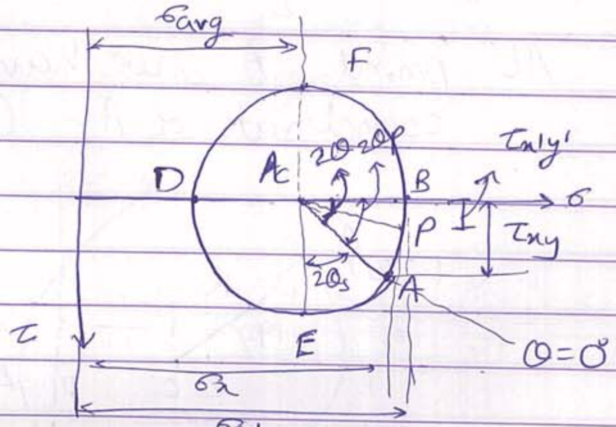
② Direction of rotation remains the same. Anticlockwise rotation in physical plane, leads to anticlockwise rotation in Mohr plane.

Mohr's circle : Construction steps

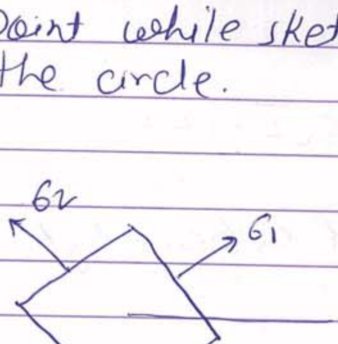
① Estabish coordinate system (σ +ve right, τ +ve down)
② Plot center $C = (\sigma_x + \sigma_y)/2$
③ Locate reference point A (σ_x, τ_{xy})
④ Connect point A with point C to get radius R
⑤ Note that $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
⑥ Sketch the circle.

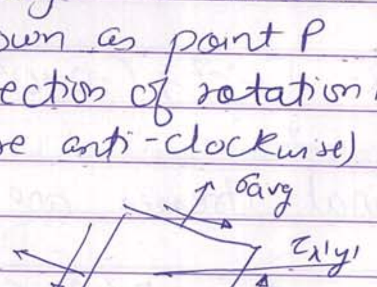


Original element (Point A)
Original state of stress refers to point A. Note, we had established this point while sketching the circle.



- At angle θ in the physical plane, i.e. angle 2θ in Mohr Plane
- Shown as point P
- Direction of rotation is same (here anti-clockwise)

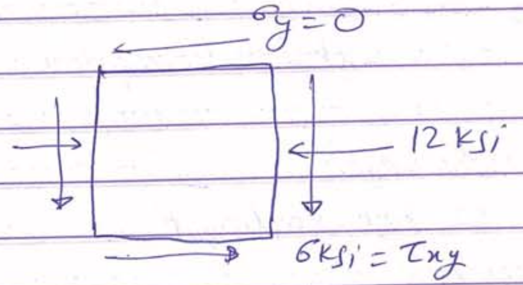




Principal stresses
- σ_1 & σ_2 - coordinates of points B & D, where circle intersects with σ axis, $\tau = 0$
- The stresses act on physical plane defined by angles θ_{p_1} & θ_{p_2} i.e. $2\theta_{p_1}$ & $2\theta_{p_2}$ in Mohr plane.
- It is measured from reference line CA to CB.

Max Shear
- σ_{avg} & max $\tau_{x'y'}$ coordinates of point E & F.
- Angles θ_{s_1} & θ_{s_2} i.e. $2\theta_{s_1}$ & $2\theta_{s_2}$ in Mohr plane. As shown, it is measured from the reference line CA to CE.

(70) Problem Example

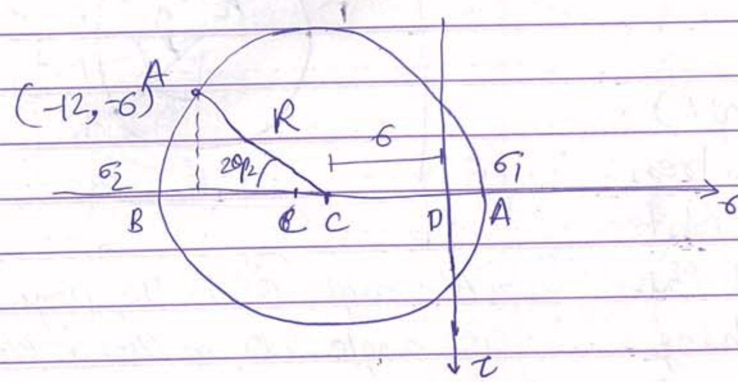


- Due to applied loading, state of stress at A is shown.
- Determine principal stresses at the same point A.

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-12 + 0}{2} = -6. \text{ (center of circle)}$$

At point A, we have τ_{xy} & σ_x

coordinate of A = $(\sigma_x, \tau) = (-12, -6)$



$$R = \sqrt{(-6)^2 + (-12+6)^2} = \sqrt{6^2 + 6^2} = 6\sqrt{2} = 8.49$$

Principal stresses are extreme right & extreme left.

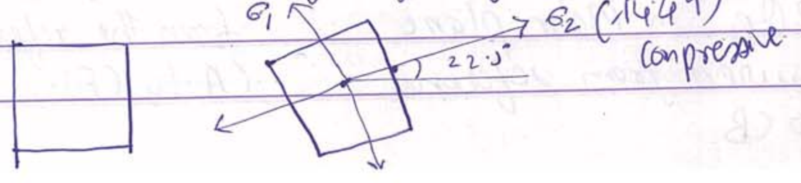
$\sigma_1 = DA = 8.49 - 6 = 2.49$ ksi

$\sigma_2 = DB = -6 - 8.49 = -14.49$ ksi.

$$2\theta_p = \tan^{-1}\left(\frac{6}{12-6}\right) = \tan^{-1}(1) = 45^\circ$$

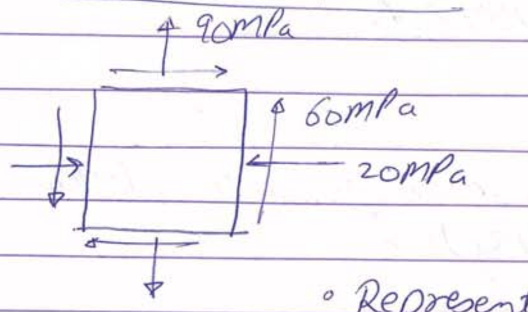
$\therefore \theta_p = 22.5^\circ$.

The face of element that bears load of -14.5 ksi must be at an angle of 22.5°.



$\theta_{p_1} = 90° - 22.5° = 67.5°$

(71) Problem Example 2



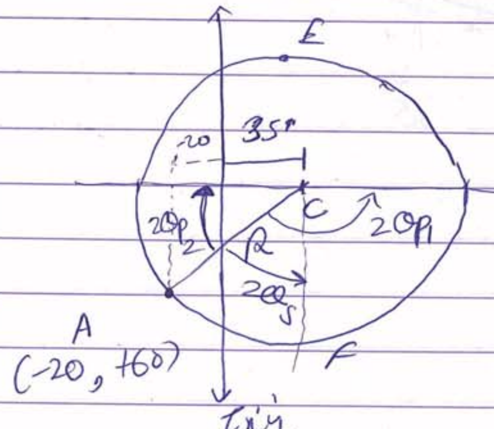
- State of stress at failure point
- Represent state of stress in terms of principal stress. Show diagram.
- Represent state of stress in terms of maximum in plane shear & associated average normal stress. Show diagram.

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20+90}{2} = 35 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = \sqrt{\left(\frac{110}{2}\right)^2 + 60^2} = \sqrt{55^2 + 60^2} = 81.39 \text{ MPa}$$

$$\sigma_1 = \sigma_{avg} + R = 81.39 + 35 = 116.39 \text{ MPa}$$

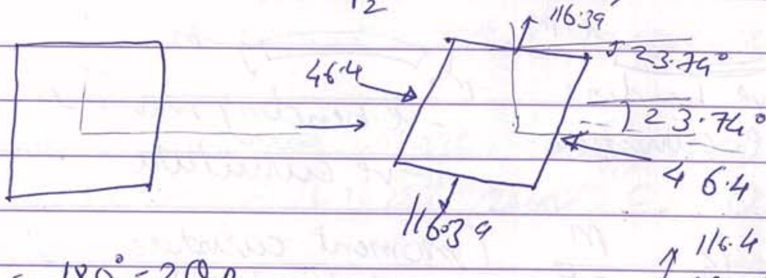
$$\sigma_2 = \sigma_{avg} - R = 35 - 81.39 = -46.39 \text{ MPa}$$

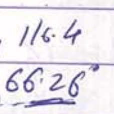


clockwise rotation.

$$\tan 2\theta_{p_2} = \frac{60}{(35+20)} = \frac{60}{55}$$

$$\Rightarrow \theta_{p_2} = 23.74°, \text{ clockwise}$$





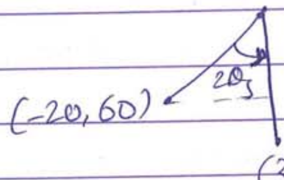
$$\tan 2\theta_{p_1} = 180° - 2\theta_{p_2}$$

$$2\theta_{p_1} = 132.52° \quad \theta_{p_1} = 66.26°$$

For maximum inplane shear. These are points top & bottom.

At E & F $\sigma = \sigma_{avg} = 35$ MPa

and $\tau_{xy\,max}$ = Radius = 81.39 MPa.



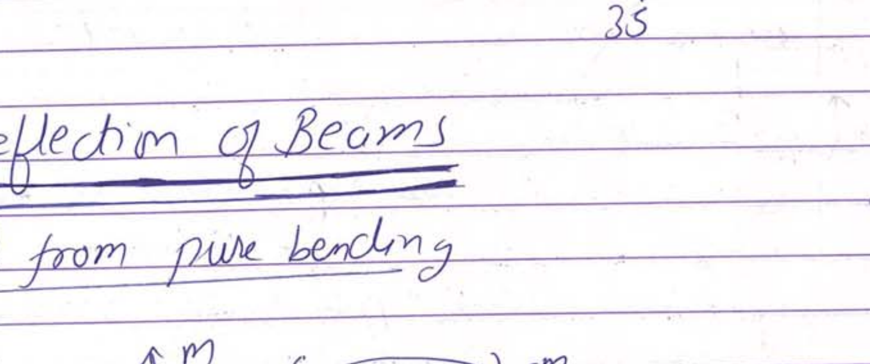
We move in anti-clockwise direction

~~tan $2\theta_s$ =~~

We know shear plane & principal plane are 45° away.

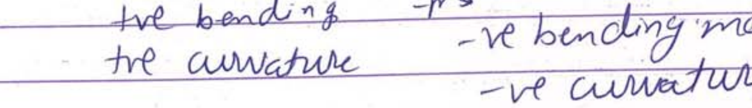
~~So~~ Principal plane $\theta_{p_2} = 23.74$.

So, $\theta_s = 45 - 23.74 = 21.26$ ↑anti-clockwise



(72) Deflection of Beams

Recall from pure bending



+ve bending
+ve curvature

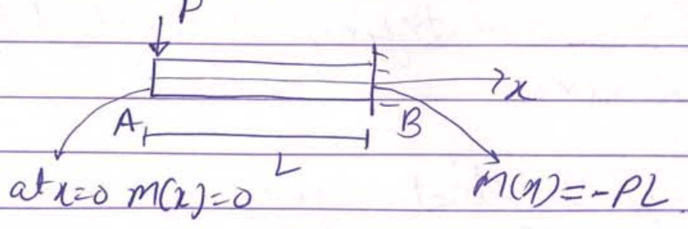
-ve bending moment
-ve curvature

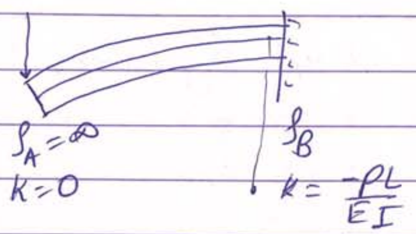
$k = 1/\rho = \frac{M}{EI}$ (moment curvature relationship)

$\sigma = -E k y$, $|\sigma| = \left|\frac{My}{I}\right|$

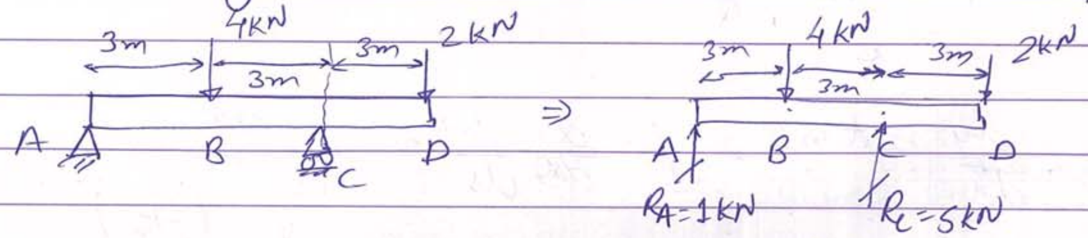
- For a general case of loading, bending moment M varies from one section to another.

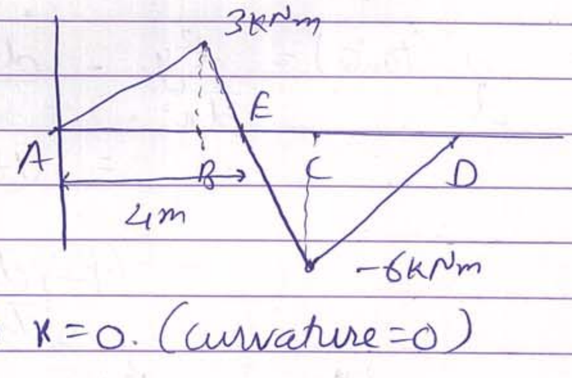
$$\therefore \; K = \frac{1}{\rho} = \frac{M(x)}{EI}$$



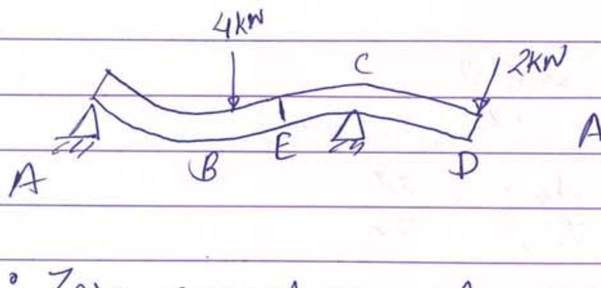


K varies along the beam. Consider another example





At A, E & D, K is 0.
So, the beam looks as →



A, E, D $K=0$. (curvature = 0)

- Zero curvature at points of zero bending moment.
- For ABE, M is +ve, so K is +ve, concave up.
- for ECD, M is -ve, so K is -ve, concave down.

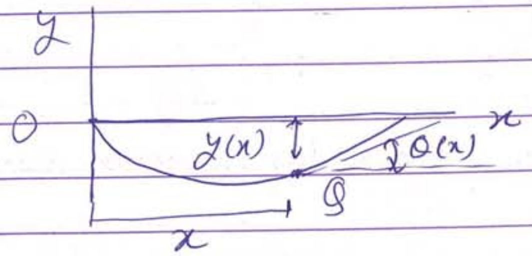
- K will be absolute maximum where bending moment is maximum. So, here it is at point C. Max curvature at C.
- Curvature helps us establish a sense of shape of beam.
- We need an equation of the beam shape (elastic curve) to find deflections and slope.

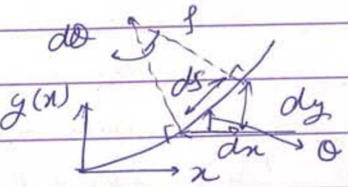
(73) Second order Method

We are trying to find deflection y and the x-axis.

eqn of elastic curve

Deflected shape of beam is known as elastic curve.





$$\frac{dy}{dx} = \tan\theta, \quad \frac{1}{\rho} = \cancel{\frac{dy}{d\theta}}\frac{d\theta}{ds} = \frac{d\theta}{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\frac{d}{dx}(\tan\theta) = \frac{d^2y}{dx^2} = \frac{d(\tan\theta)}{d\theta} \times \frac{d\theta}{dx}$$

$$= (1+\tan^2\theta) \times d\theta/dx$$

$$= \left(1+\left(\frac{dy}{dx}\right)^2\right) \times \frac{d\theta}{dx}$$

$$\cancel{\frac{d\theta}{dx}} \Rightarrow \frac{d^2y}{dx^2} = \left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{d^2y/dx^2}{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$$

$$K = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}} = \frac{d^2y/dx^2}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

$$\simeq d^2y/dx^2$$

$$\boxed{K = \frac{1}{\rho} = \frac{d^2y/dx^2}{\left(1+(dy/dx)^2\right)^{3/2}}}$$

→ excat expression for curvature-deflection relationship.

For small angles of rotation

$$k = \frac{1}{\rho} = \frac{d^2y/dx^2}{\left(1 + \left(dy/dx\right)^2\right)^{3/2}} \approx \frac{d^2y}{dx^2}.$$

$$k = \frac{1}{\rho} = \boxed{\frac{M(x)}{EI} = \frac{d^2y}{dx^2}}$$ (valid for small deflection)

deflection → y. Relationship b/w deflection(y) & bending moment (M)

$$\Rightarrow \boxed{EI \frac{d^2y}{dx^2} = M(x)}$$

Since we start with second order derivative of deflection, this method is known as second order method.

$$\Rightarrow EI \frac{dy}{dx} = \int_0^x M(x)\, dx + C_1$$

(Assume: Beam is prismatic, $I(x) = I$)

For small angles, $\frac{dy}{dx} = \tan\theta = \theta(x)$

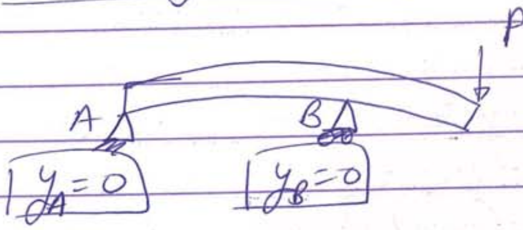
$$\Rightarrow EI\,\theta(x) = \int_0^x M(x)\, dx + C_1$$ → Gives slope.

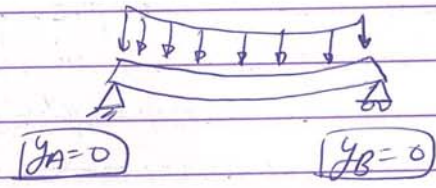
$$\Rightarrow EIy = \int_0^x \left[\int_0^x M(x)\, dx + C_1\right] dx + C_2$$ → Gives deflection

C_1 & C_2 are constant of integration & we get using boundary condition.

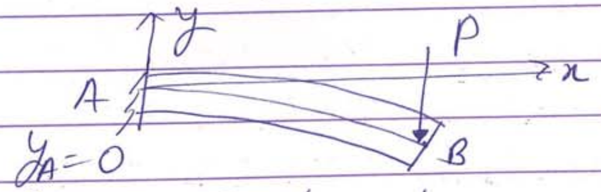
$\boxed{y(x) \text{ is eq}^n \text{ of elastic curve}}$

Boundary Conditions





Overhanging beam   Simply supported beam

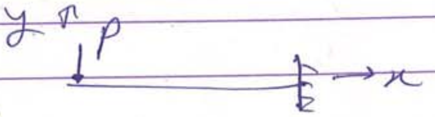


$y_A = 0$

$\theta_A = 0$ Cantilever beam

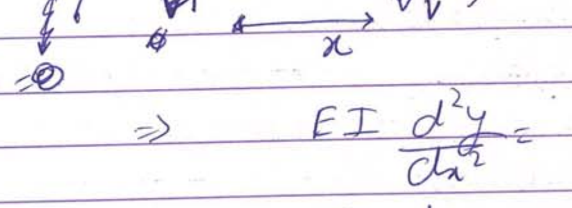
- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

(74) 2^{nd} order Methods (Problem)



$$EI\frac{d^2y}{dx^2} = M(x)$$

Using method of sections we get moment.



$$M = -Px$$

$$\Rightarrow \quad EI\frac{d^2y}{dx^2} = -Px$$

$$\Rightarrow \quad EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1$$

$$\Rightarrow \quad EI\,y = -\frac{Px^3}{6} + C_1x + C_2$$

At end B, ie $x = L$, $dy/dx = 0$ & $y = 0$.

So, at $x = L$, $-\frac{P}{2}L^2 + C_1 = 0$ $\left(dy/dx \text{ @ } x = L\right)$

$C_1 = PL^2/2$

at $x = L$, $y_B = 0$, $-\frac{PL^3}{6} + \frac{PL^2}{2} \times L + C_2 = 0$

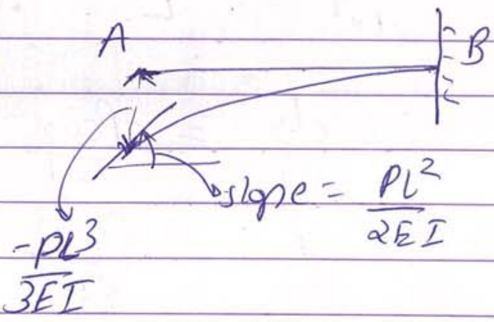
$= C_2 = -PL^3/3$

$EI\,y(x) = -\frac{Px^3}{6} + \frac{PL^2}{2}x - \frac{PL^3}{3}$ (Deflection)

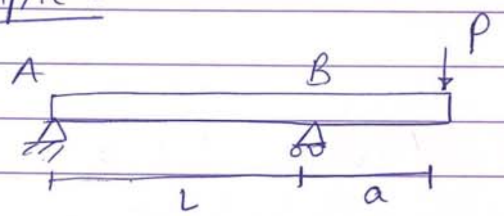
$EI\frac{dy}{dx} = \frac{-Px^2}{2} + \frac{PL^2}{2}$ (slope)

Now, we need deflection and slope at A. ie @ $x = 0$

$$\boxed{\begin{aligned} \left.\frac{dy}{dx}\right|_{x=0} &= \frac{PL^2}{2EI} \\ y_A &= \frac{-PL^3}{3EI} \end{aligned}}$$



example 2

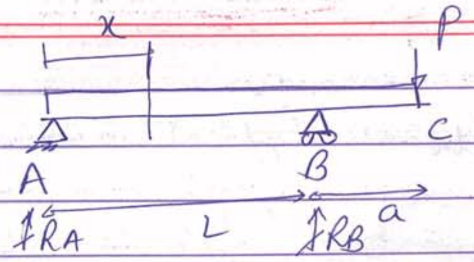


$I = 300 \times 10^6\ mm^4$

$E = 200\ GPa$ $P = 200\ kN$

$L = 4.5m$ $a = 1.2m$.

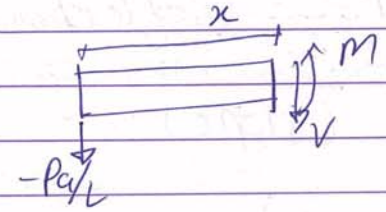
For the portion AB of the overhang beam (a) derive eq^n for elastic curve (b) find maximum deflection (c) evaluate Y_{max}.



$R_A + R_B = P$; $M@A = 0$

$$R_B \times L = P \times (a+L)$$
$$R_B = P(a+L)/L$$
$$R_A = P - \frac{P(a+L)}{L} = -Pa/L$$

at distance x



$$\boxed{M = \frac{-Pax}{L}}$$
$$0 < x < L$$

$$EI\frac{d^2y}{dx^2} = \frac{-Pax}{L}$$

$$\Rightarrow EI\frac{dy}{dx} = \frac{-Pax^2}{2L} + C_1$$

$$\Rightarrow EIy = \frac{-Pax^3}{6L} + C_1x + C_2$$

~~Boundary~~ This is valid from $x=0$ to $x=L$.

@ A & @ B $y=0$.

$x_A = 0 \Rightarrow C_2 = 0$; $x_B = 0 \Rightarrow \frac{-PaL^2}{6} + C_1L = 0$

$$\Rightarrow C_1 = PaL/6$$

$\Rightarrow$ ~~$EI\,dy/dx =$~~

$\therefore EIy = \frac{-Pax^3}{6L} + \frac{PaLx}{6}$ → eqⁿ of elastic curve

Max deflection → (for max deflection $dy/dx = 0$)
$dy/dx = 0$.

$$\Rightarrow \quad \frac{-Pax^2}{2L} + \frac{PaL}{6} = 0$$

$$\Rightarrow \quad x^2 = \frac{L^2}{3} \Rightarrow \boxed{x = \frac{L}{\sqrt{3}}}$$

We get max deflection at $x = L/\sqrt{3}$.

$$EI\, y_{max} = \frac{-PaL^3}{6L \times 3\sqrt{3}} + \frac{PaL^2}{6 \times \sqrt{3}}$$

Substituting values

$$200 \times 10^3 \times 300 \times 10^6 \times y = -\frac{200 \times 10^3 \times 1.2 \times 10^3 \times (4.5 \times 10^3)^2}{18\sqrt{3}} + \frac{200 \times 10^3 \times 1.2 \times 10^3 \times (4.5 \times 10^3)^2}{6\sqrt{3}}$$

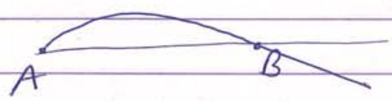
$$6 \times 10^{16} \times y = \frac{-200 \times 1.2 \times 4.5 \times 4.5 \times 10^{12}}{18\sqrt{3}} + \frac{200 \times 1.2 \times 4.5 \times 4.5}{6\sqrt{3}} \times 10^{12}$$

$$60y = \frac{200 \times 1.2 \times 4.5 \times 4.5}{6\sqrt{3}} \left(-\frac{1}{3} + 1 \right)$$

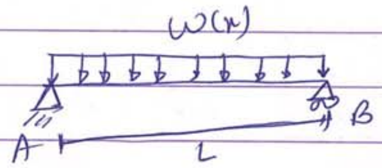
$$y = \frac{20 \times 1.2 \times 4.5 \times 4.5 \times 2}{6 \times 6 \times 3\sqrt{3}} = 5.19\,mm.$$

It is +ve, so it is in upwards direction.

Our beam looks as



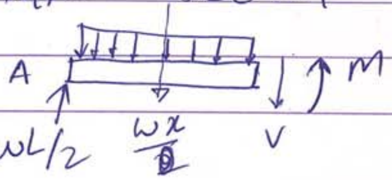
Example 3



Total load $= wL$

$R_A = wL/2$, $R_B = wL/2$

We take section



$$M - \frac{wL}{2}x + wx \times \frac{x}{2} = 0$$

$$M = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = M(x)$$

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

$$EI\, y(x) = \frac{wLx^3}{12} - \frac{wx^4}{24} + C_1x + C_2$$

@ $x = 0$ & @ $x = L$, $y = 0$.

$$C_2 = 0$$

@ $\frac{wL^4}{12} - \frac{wL^4}{24} + C_1L = 0$

$$C_1 = -\frac{w_0L^3}{24} \cancel{-\frac{wL^3}{12}} =$$

y_{max} occurs at $L/2$.

$$EI\, y_{max} = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x$$

@ $x = L/2$

$$EI\, y_{max} = \frac{wL \times L^3}{12\times 8} - \frac{wL^4}{24\times 16} - \frac{wL^3 \times L}{24 \times 2}$$

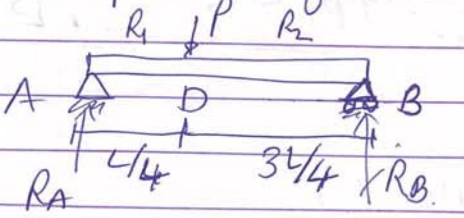
$$EI\, y_{max} = \frac{wL^4}{24}\left(\frac{1}{4} - \frac{1}{16} - \frac{1}{2}\right) = \frac{wL^4}{48}\left(\frac{1}{2} - \frac{1}{8} - 1\right)$$

$$EI\, y_{max} = \frac{-5wL^4}{8 \times 48} =$$

$$\boxed{y_{max} = \frac{-5}{384}\frac{wL^4}{EI}}$$

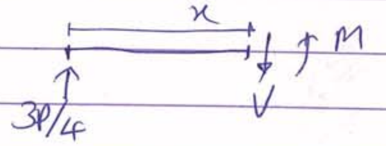
Example 4

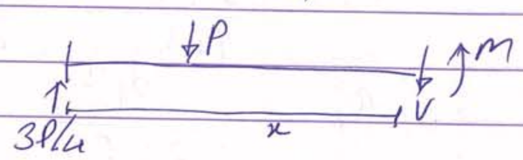
- Find slope & Deflection @ D.



$R_A = 3P/4 \qquad R_B = P/4$

$R_1: \quad 0 < x < L/4$ $\qquad\qquad$ $R_2: \quad x > L/4$





$M = 3Px/4$ $\qquad\qquad$ $M = \frac{3Px}{4} - P\left(x - \frac{L}{4}\right)$

$EI\frac{d^2y}{dx^2} = \frac{3Px}{4}$ $\qquad\qquad$ $EI\frac{d^2y}{dx^2} = \frac{3Px}{4} - P\left(x - \frac{L}{4}\right)$

$EI\,\theta_1 = \frac{3Px^2}{8} + C_1$ — (1) $\qquad$ $EI\theta_2 = \frac{-Px^2}{8} + \frac{PLx}{4} + C_3$ — (3)

$EI\,y_1 = \frac{3Px^3}{8} + C_1x + C_2$ — (2) $\qquad$ $EIy_2 = \frac{-Px^3}{24} + \frac{PLx^2}{8} + C_3x + C_4$ — (4)

@ $x=0$, $y_1=0$ $\qquad\qquad$ @ $x=L$, $y_2=0$

@ $x = L/4$ — $y_{L/4}$ is same for both

$\left[\, x = L/4, \quad \theta_1 = \theta_2 \ \& \ y_1 = y_2 \,\right]$

Slope compatibility $\qquad$ Displacement compatibility

$C_2 = 0$ @ $x=0$ $\qquad\qquad$ $C_2 = 0$

@ $x=L$ $\quad y_2=0$ $\qquad$ $\frac{-PL^3}{24} + \frac{PL^3}{8} + C_3L + C_4 = 0$

@ $x = L/4$ $\qquad$ $\frac{PL^3}{8\times64} + C_1\times\frac{L}{4} = \frac{-PL^3}{24\times64} + \frac{PL^3}{8\times16} + C_3\times\frac{L}{4} + C_4$

@ $x = L/4$ $\qquad$ $\frac{3PL^2}{8\times16} + C_1 = \frac{-PL^2}{8\times16} + \frac{PL^2}{16} + C_3$

We get

$$C_1 = \frac{-7PL^2}{128}\ ;\ C_2 = 0\ ;\ C_3 = \frac{-11PL^2}{128}\ ;\ C_4 = \frac{PL^3}{384}$$

So, slope and deflection at point D.

$$EI\theta_1 = \frac{3Px^2}{8} - \frac{7PL^2}{128}\ ,\quad x = L/4$$

$$EI\theta_1 = \frac{3PL^2}{128} - \frac{7PL^2}{128}$$

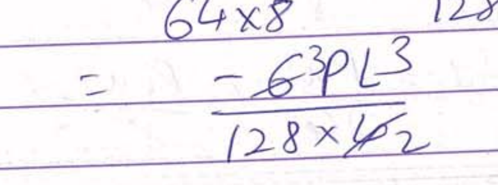
$$\boxed{\theta_D = \frac{-PL^2}{32EI}}$$

$$EIy_1 = \frac{Px^3}{8} - \frac{7PL^2x}{128}\ ,\quad x = L/4$$

$$EIy = \frac{PL^3}{64\times 8} - \frac{7PL^3}{128\times 4} = \frac{PL^3}{128\times 4}\left(\frac{1}{2} - 1\right)$$

$$= \frac{-\cancel{6}^{3}PL^3}{128\times\cancel{4}_{2}} = \frac{-3PL^3}{256}$$

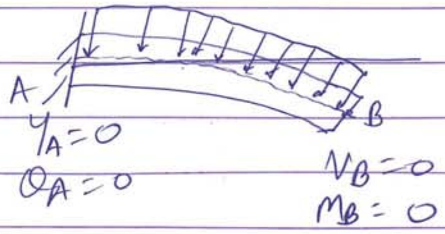
$$\boxed{y_D = \frac{-3PL^3}{256EI}}\ @\ \cancel{y}\ x = L/4$$

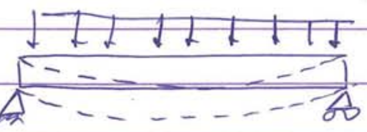


If it was max deflection, slope would be 0. So, here it is not max deflection.

(75) 4^{th} order Method

Elastic curve & deflection (4^{th} order Method)





- For a beam subjected to a distributed load

$$\frac{dM}{dx} = V(x) \; ; \quad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x) \rightarrow \text{from SFD \& BMD.}$$

$\Rightarrow$ Eq^n for beam displacement becomes

$$EI\frac{d^2y}{dx^2} = M$$

$$\Rightarrow \boxed{EI\frac{d^4y}{dx^4} = -w(x)}$$

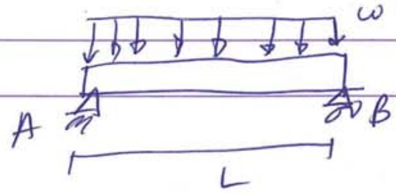
- So, using inter relations we get 4^{th} order displacement derivative related to distributed force.

Integrating 4 times yields

$$EI\,y(x) = -\int dx \int dx \int dx \int dx\, w(x) + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

Q) Find elastic curve eq^n & max deflection of beam



$$EI\frac{d^4y}{dx^4} = -w(x) = -w$$

$$EI\ y(x) = -\int dx\int dx\int dx\int dx\, w + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$$

$$EI\,y = -\frac{wx^4}{24} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$$

@ $x=0$ $y=0$ $\Rightarrow C_4 = 0$

@ $x=L$ $y=0$ $\Rightarrow -\frac{wL^4}{24} + \frac{C_1L^3}{6} + \frac{C_2L^2}{2} + C_3L = 0$

$$\frac{C_1L^2}{6} + \frac{C_2L^2}{2} + C_3 = \frac{wL^3}{24}$$

@ $x=0$ $M=0$ $\quad -\frac{wx^2}{2} + C_1x + C_2 = 0$

$\Rightarrow C_2 = 0$

@ $x=L$ $M=0$ $\quad -\frac{wL^2}{2} + C_1L = 0$

$C_1 = wL/2.$

$$\frac{wL^3}{\cancel{24}12} + C_3 = \frac{wL^3}{24}$$

$$C_3 = -\frac{wL^3}{24}$$

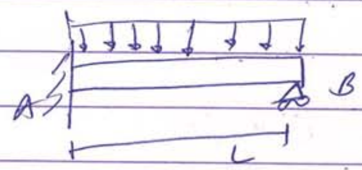
$$EI\,y = -\frac{wx^4}{24} + \frac{wLx^3}{12} - \frac{wL^3x}{24}$$

Max @ $x = L/2$

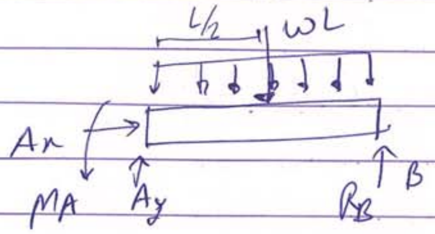
$$EI\,y_{max} = -\frac{5}{348}wL^4$$

$$y_{max} = \frac{-5}{348}\frac{wL^4}{EI}$$

(76) Deflection of Indeterminate Beams (Theory + Problems)



propped cantilever, statically indeterminate to degree 1.
(3 unknowns, 2 eq^{n}s)
4 unknowns but ~~A_y~~ $A_x = 0$.



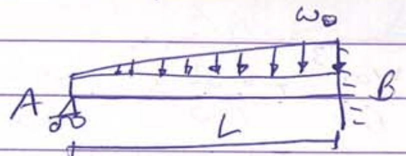
$$EI\,y = \int dx \int M(x)\,dx + C_1 x + C_2$$

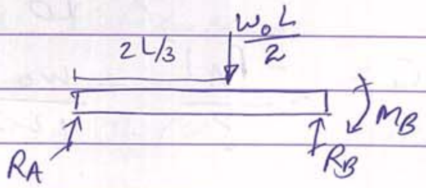
The eq^n introduces two unknowns but provides 3 additional eq^{n}s from the boundary conditions (used to solve for C_1, C_2, R_B):

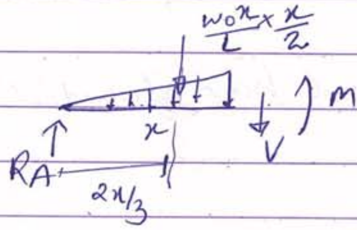
$y = 0$ @ $x = 0$; $y = 0$ @ $x = L$, $dy/dx = 0$ @ $x = 0$ $\xrightarrow{\text{so}}$ 0.

Example

Find reaction at A, derive eq^n for elastic curve, & find slope at A.







$$M = R_A \times x - \frac{w_0 x^2}{2L} \times \frac{2x}{3}$$

$$M = R_A \times x - \frac{w_0 x^3}{6L}$$

$$EI\frac{d^2y}{dx^2} = M(x) = R_A x - \frac{w_0 x^3}{6L}$$

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{w_0 x^4}{24L} + C_1$$

$$EI\,y = \frac{R_A x^3}{6} - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

@ $x = 0$, $y = 0$ @ $x = L$, $y = 0$ @ $x = L$ $dy/dx = 0$.

and @

$$\frac{R_A L^2}{2} - \frac{w_0 L^4}{24L} + C_1 = 0 \quad \left(dy/dx = 0 \ @ \ x = L \right)$$

$$0 + 0 + 0 + C_2 = 0 \quad (y = 0 \ @ \ x = 0)$$

$$\frac{R_A L^3}{6} - \frac{w_0 L^5}{120L} + C_1 L = 0 \quad (y = 0 \ @ \ x = L)$$

$$\frac{R_A L^2}{6} - \frac{w_0 L^3}{120} + C_1 = 0$$

$$\frac{R_A L^2}{2} - \frac{w_0 L^3}{24} + C_1 = 0$$

$$\Rightarrow \quad \frac{R_A L^2}{6} - \frac{w_0 L^3}{120} - \frac{R_A L^2}{2} + \frac{w_0 L^3}{24} = 0$$

$$R_A \left(\frac{-1}{3} \right) L^2 = w_0 L^3 \left(\frac{1}{120} - \frac{1}{24} \right)$$

$$\frac{-R_A}{3} = \frac{w_0 L}{120} \times (-4) \Rightarrow$$

$$R_A = \frac{w_0 L}{10}$$

$$C_1 = \frac{-R_A L^2}{6} + \frac{w_0 L^3}{120} = \frac{-w_0 L^3}{60} + \frac{w_0 L^3}{120} = \frac{-w_0 L^3}{120}$$

So, without using eqm equations we have found R_A simply using deflection steps.

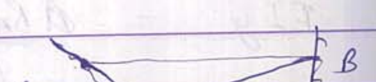
Eqn of elastic curve

$$EIy = \frac{R_A x^3}{6} - \frac{w_0 x^5}{120L} + C_1 x$$

$$= \frac{w_0 L x^3}{60} - \frac{w_0 x^5}{120L} - \frac{w_0 L^3 x}{120}$$

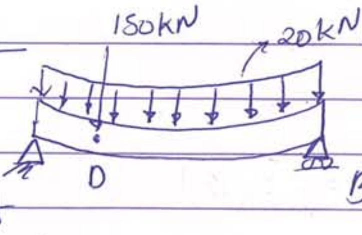
Slope at A; $\quad EI \frac{dy}{dx} = C_1 \quad (x = 0)$

$$\left. \frac{dy}{dx} \right|_{x=A} = \frac{-w_0 L^3}{120 EI}$$



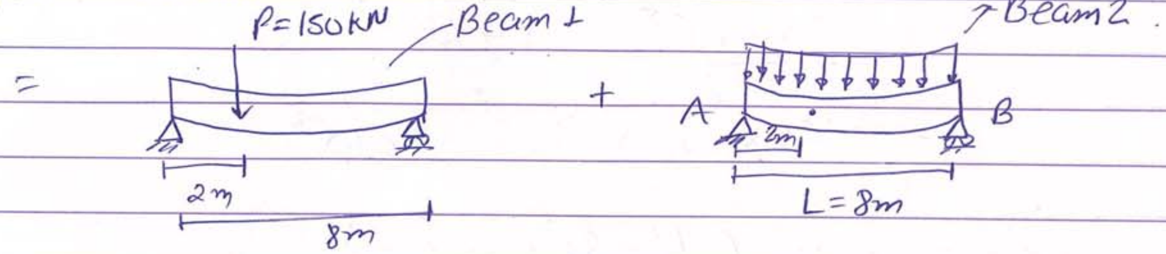
(77) Beam deflection using method of Superposition.

Problem



Find slope & deflection at D
Given $EI = 100\ MN \cdot m^2$.

As long as we have small deflection we can split beam as →



Principle of superposition:

- Deformation of beams subjected to combinations of loading may be obtained as a linear combination of the deformations from the individual loadings.

$$\Delta_D = \Delta_{D,Beam1} + \Delta_{D,Beam2}.$$

This process is facilitated by use of certain tables ~~that~~ known as deflection table.
From table we need to find deflection at D for both cases & add them up.

$$\Delta_{D,B1} = \frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) = \frac{-150\times10^3 \times \cancel{6}\times 2}{\cancel{6}\times \cancel{8}^{4} \times 100\times10^6} \times (8^2 - 6^2 - 2^2)$$

$$= \frac{-150\times10^{-5}}{4}(24) = -\frac{3600\times10^{-5}}{4} = -36\ mm = -9mm$$

$$\Delta_{D,B2} = \frac{-20\times10^3\times 2}{24\times100\times10^6}(8^3 - 2\times8\times2^2 + 2^3)$$

$$= \frac{-\cancel{20}^{10}\times10^3\times\cancel{2}\times10^{-8}}{\cancel{24}\ \cancel{12}\ 6}(512 - 64 + 8)$$

$$= \frac{-10^{-4}}{6}(456) = -76\times10^{-4}\times10^3\ mm = -7.6\ mm$$

$$\Delta = -9 - 7.6 = -16.60mm.$$

Similarly we can calculate slope.

$$\theta_{D,B_1} = \frac{-Fb}{6LEI}\left(L^2-b^2-3x^2\right)$$

$$= \frac{-150\times10^3\times \cancel{6}}{\cancel{6}\times 8\times 100\times10^6}\left(8^2-6^2-3\times4\right)$$

$$= \cancel{\frac{-150\times10^{-5}\times(36)}{8}} = \cancel{-675\times10^{-5}}$$

$$= \frac{-150\times10^{-5}\times(64-36-12)}{8}$$

$$= \frac{-150\times10^{-5}\times16}{8} = -3\times10^{-3}.$$

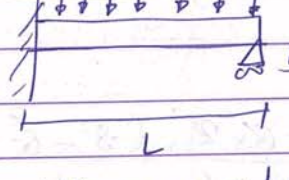
$$\theta_{D,B_2} = \frac{-w}{24EI}\left(L^3-6Lx^2+4x^3\right)$$

$$= \frac{-20\times10^3}{24\times10^8}\left(8^3-6\times8\times4+4\times \cancel{64}\,8\right)$$

$$= \cancel{-20\times10^{-3}\times576/24} \quad -2.93\times10^{-3}$$

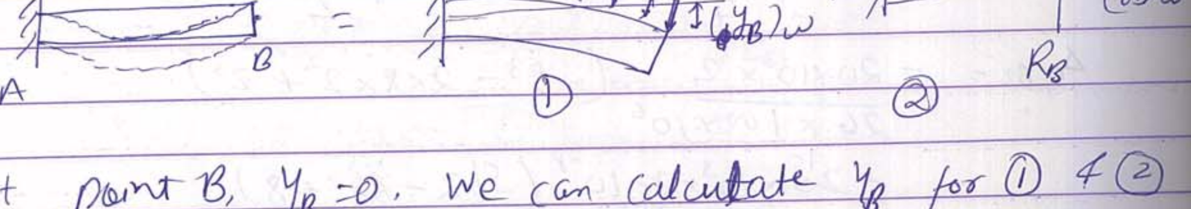
$$\theta_D = -3\times10^{-3} - 2.93\times10^{-3} = -5.93\times10^{-3}\ \text{rad}$$

* How can we apply method of superposition to statically indeterminate beams



Degree of indeterminacy = 1.
Need to choose one redundant.

we can choose $y_B = 0$.

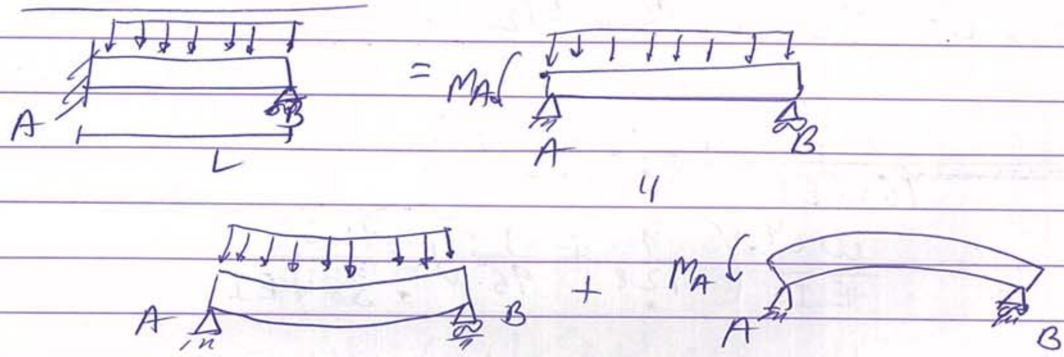


At point B, $y_B = 0$. We can calculate y_B for ① & ②

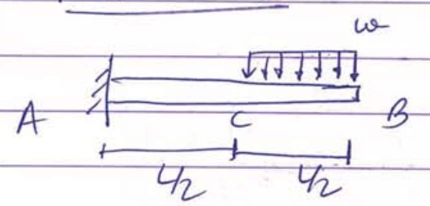
Compatibility eq^n → $(y_B)_w + (y_B)_R = 0.$

- Designate one of the reactions as the redundant and eliminate or modify the support.
- ⇒ Note that you must ensure that redundant chosen does not make structure unstable
- Determine beam deformations without redundant reaction
- Treat redundant reaction as an unknown load which, together with the other (i.e. applied) loads, must produce deformations compatible with the original supports

Alternate choice

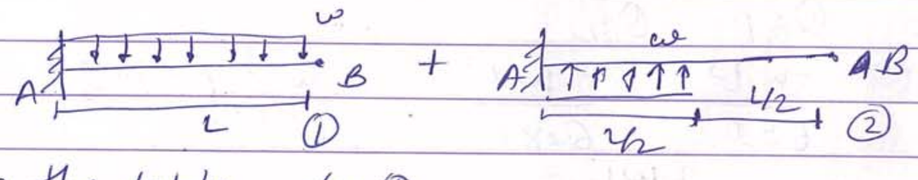


(78) Problems



- Find deflection and slope at B.

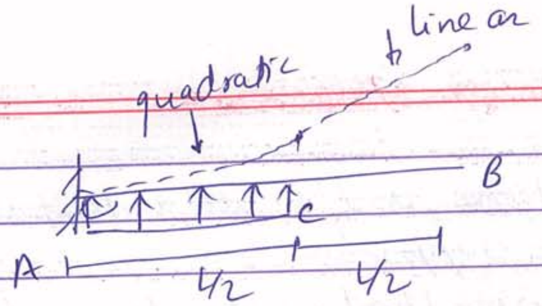
We can write our beam as ⇒



From the table, for ①

$$\delta_{B,1} = \frac{-wL^4}{8EI}, \quad \theta_{B,1} = \frac{-wL^3}{6EI}$$

Now for Beam ②



It has quadratic deflection till L/2 & linear afterwards.

Now, for L/2.

$$\delta_{B,2,L/2} = \frac{wL^4}{16\times 8EI} \qquad \theta_{B,2,L/2} = \frac{wL^3}{6\times 8EI}$$

Slope at C = slope at B. (As it is linear)

$$\delta_{B,2,L} = \delta_C + \theta_C \times L/2$$

$$= \frac{wL^4}{16\times 8EI} + \frac{wL^3}{6\times 8EI}\times\frac{L}{2}$$

$$= \frac{wL^4}{EI}\left(\frac{1}{128}+\frac{1}{96}\right) = \frac{7wL^4}{384EI}$$

So, $\delta_B = \delta_{B,1} + \delta_{B,2}$

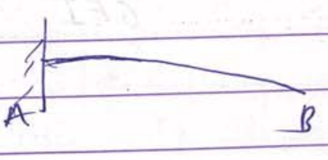
$$= \frac{-wL^4}{8EI} + \frac{7wL^4}{384EI}$$

$$= \frac{wL^4}{8EI}\left(-1+\frac{7}{48}\right) = \frac{-41wL^4}{384EI}$$

$$\theta_B = \theta_{B,1} + \theta_{B2}$$

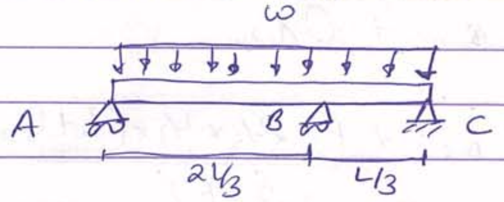
$$= \frac{-wL^3}{6EI} + \frac{wL^3}{6\times 8EI} = \frac{wL^3}{6EI}\left(-1+\frac{1}{8}\right)$$

$$= \frac{-7wL^3}{48EI}$$



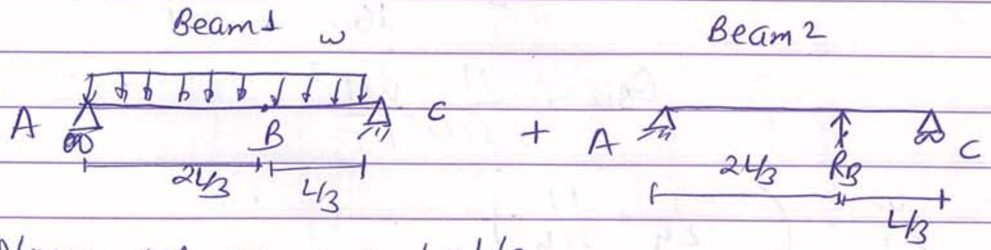
Problem 2

Find reactions at each support & slope at A.



R_A, R_B, R_C

We can take B as redundant and split our beam as →



Now, we can use tables.

$$Y_B = 0. = Y_{B_1} + Y_{B_2} = 0 \rightarrow \text{Compatibility eq}^n.$$

$$y_{B,1} = \frac{-w \times 2L/3}{24EI}\left(L^3 - 2 \times L \times \frac{4L^2}{9} + \frac{8L^3}{27}\right)$$

$$= \frac{-wL}{36EI}\left(\cancel{L^3} - \cancel{\frac{8L^3}{9}} \frac{3L^3}{2\cancel{7}} + \frac{8L^3}{27}\right)$$

$$= \frac{-11wL^4}{36 \times 27 EI}$$

$$Y_{B_2} = \frac{P \times L/3 \times 2L/3}{6L\,EI}\left(L^2 - \frac{L^2}{9} - \frac{4L^2}{9}\right) = \frac{PL}{27EI}\left(\frac{4L^2}{9}\right)$$

$$= \frac{4PL^3}{243EI}$$

$$Y_{B,1} + Y_{B,2} = 0 \Rightarrow \frac{4PL^3}{243\cancel{EI}} = \frac{11wL^4}{36 \times 27\cancel{EI}}$$

$$R_B = P = \frac{11 wL \times 243}{4 \times 36 \times 27} = 0.6875\, wL.$$

$$R_A + R_C = 0.3125\, wL$$

$$\circlearrowleft M_A = 0 \qquad -\frac{\cancel{L}}{2} \times wL + \frac{2\cancel{L}}{3} \times \frac{11}{16} wL + \cancel{L} \times R_C = 0$$

$$R_C = \frac{wL}{2} - \frac{11wL}{24} = \frac{wL}{24} = 0.0416\, wL.$$

R

$\Rightarrow R_A = 0.271\,wL$

Slope at A. $\theta_A = \theta_{A,1} + \theta_{A,2}$

$$\theta_{A,1} = \frac{-wL^3}{24EI}, \quad \theta_{A,2} = \frac{+P\times 2L/3 \times L/3 \times (L+L/3)}{6LEI}$$

$$\theta_{A,2} = \frac{P\times 2L^2}{27EI}\times\frac{4L}{3}$$

$$\theta_{A,2} = \frac{11}{164}\frac{wL\times 4L^2}{81EI}$$

$$\theta_{A,2} = \frac{11}{324}\frac{wL^3}{EI}$$

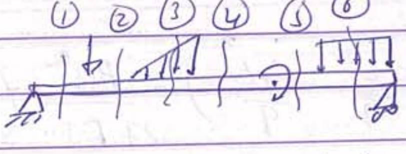
$$\theta = \frac{wL^3}{EI}\left(\frac{-1}{24}+\frac{11}{324}\right)$$

$$= 0.0077\frac{wL^3}{EI}$$

(79) Beam Deflection using singularity Functions

Amazing method for solving deflection & slope.

- Issues in 2^{nd} order, 4^{th} order or method of superposition



In this we will need to have many sections to write eqn's for moment then write support condition & continuity condition.
We will need 6 sections for above bar.
So, we need to know new method for solving the beam deflection problem. (using Singularity function)

Singularity Functions or Macaulay Functions
(Mathematical Rules)

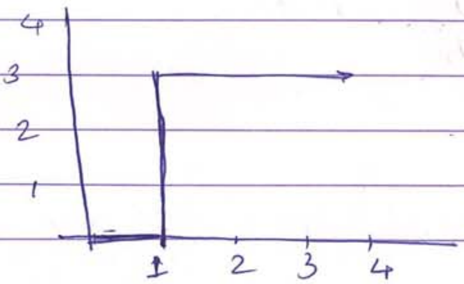
A singularity function is defined as:

[$< >$: macaulay brackets]

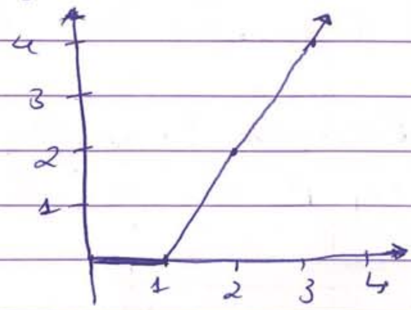
$$f(x) = \langle x-a \rangle^n$$
$$= \begin{cases} (x-a)^n & \text{for } x \geq a \\ 0 & \text{for } x < a. \end{cases} \quad \text{valid for } n \geq 0.$$

Ex:

$f(x) = 3\langle x-1 \rangle^0$



$f(x) = 2\langle x-1 \rangle^1$

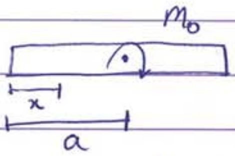
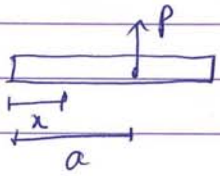


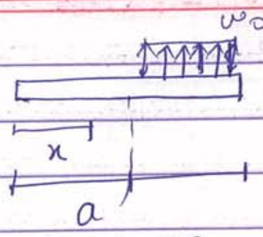
Integration rules for Macaulay functions

$$\int \langle x-a \rangle^n dx = \begin{cases} \dfrac{\langle x-a \rangle^{n+1}}{n+1} & \text{for } n \geq 0 \\ \langle x-a \rangle^{n+1} & \text{for } n < 0 \end{cases}$$

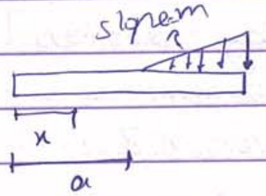
Interrelationships.

Sign convention: P, W positive upwards. M +ve clockwise

Loading	$w = w(x)$ Loading fn	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
	$w = M_0 \langle x-a \rangle^{-2}$	$V = M_0 \langle x-a \rangle^{-1}$	$M = M_0 \langle x-a \rangle^0$
	$P\langle x-a \rangle^{-1}$	$V = P\langle x-a \rangle^0$	$\dfrac{P_0 \langle x-a \rangle^1}{1}$



$w = w_0 \langle x-a \rangle^0 \rightarrow V = w_0 \langle x-a \rangle^1 \rightarrow M = \frac{w_0 \langle x-a \rangle^2}{2}$

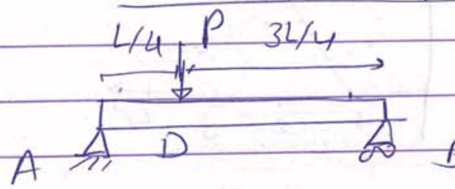


$w = m \langle x-a \rangle^1 \rightarrow V = \frac{m \langle x-a \rangle^2}{2} \rightarrow M = \frac{m}{6} \langle x-a \rangle^3$

Now, we can express each of loads in terms of moment as we know interrelationship using singularity function.

And then we use $EI \frac{d^2y}{dx^2} = M(x)$

(80) Problem Example



Find slope & deflection at D.

$R_A = 3P/4$, $R_B = P/4$

$\cancel{M(x) = -P \langle x - L/4 \rangle}$

$M(x) = \frac{3P}{4} \langle x-0 \rangle^1 - P \langle x - L/4 \rangle$

If we write for B, it will always give 0, as $x \le L$.

But we are accounting R_B, as due to roller support we are getting reaction at pin support & system is in eq^m. (Single expression)

$$EI \frac{d^2y}{dx^2} = M(x) = \frac{3P}{4} \langle x-0 \rangle^1 - P \langle x - L/4 \rangle$$

$$EI \frac{dy}{dx} = \frac{3P}{4 \times 2} \langle x-0 \rangle^2 - \frac{P}{2} \langle x - L/4 \rangle^2 + C_1$$

$$EI\, y = \frac{3P}{8 \times 3} \langle x-0 \rangle^3 - \frac{P}{2 \times 3} \langle x - L/4 \rangle^3 + C_1 x + C_2$$

Now, we find constants using boundary condition

$[x=0,\ y=0\ ;\ x=L,\ y=0]$

$$EI\times 0 = \frac{3P}{24}\underbrace{\langle 0-0\rangle^3}_{0} - \frac{P}{6}\underbrace{\langle 0-L/4\rangle^3}_{0} + C_1\times 0 + C_2$$

$\Rightarrow \quad C_2 = 0.$

$$EI\times 0 = \frac{3P}{24}\langle L-0\rangle^3 - \frac{P}{6}\langle L-L/4\rangle^3 + C_1 L$$

$$0 = \frac{3P}{24}L^3 - \frac{PL^3}{6}\times\frac{27}{64} + C_1 L.$$

$$\therefore C_1 = \frac{-7PL^2}{128}$$

Eq^n of elastic curve

$$EIy(x) = \frac{3P}{24}\langle x-0\rangle^3 - \frac{P}{6}\langle x-L/4\rangle^3 - \frac{7PL^2}{128}x$$

For deflection at $L/4$ i.e. at D.

$$EIy(x) = \frac{3P}{24}\times\frac{L^3}{64} - 0 - \frac{7PL^2}{128}\times\frac{L}{4}$$

$$EIy = \frac{PL^3}{64}\left(\frac{3}{24} - \frac{7\times 3}{8\times 3}\right) = \frac{PL^3}{64\times \cancel{24}_{4}}\left(\frac{-18}{\cancel{24}}\right)^3$$

$$y = \frac{-3PL^3}{256\,EI}$$

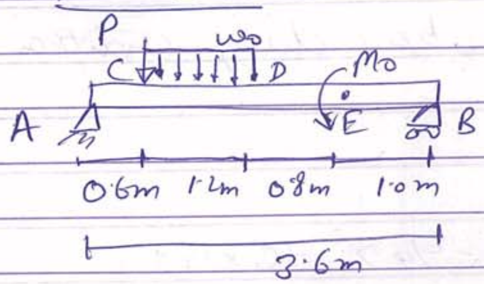
Slope at $L/4$ i.e. at D

$$EI\theta_D = \frac{3P}{8}\langle L/4-0\rangle^2 - \frac{P}{2}\langle L/4 - L/4\rangle + -\frac{7PL^2}{128}$$

$$= \frac{3P}{8}\times\frac{L^2}{16} - \frac{7PL^2}{128} = \frac{-PL^2}{32}$$

$\theta_D = -PL^2/32EI$

(81) Problem 2



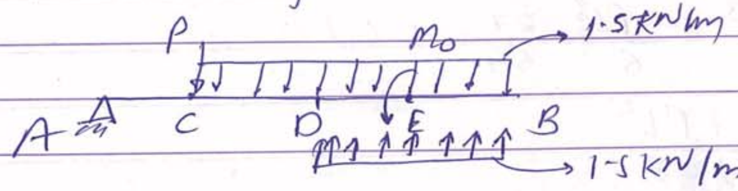
$P = 1.2\,kN$

$w_0 = 1.5\,kN/m$

$M_0 = 1.44\,kNm$

Find deflection at D.

⇒ The uniformly distributed load is not open ended here (i.e does not go till last)
So, we will first add that



First we need R_A & R_B.
$R_A = 2.6\,kN$. We don't need R_B.

$$M(x) = 2.6\langle x-0\rangle^1 - \cancel{1.5}\,1.2\langle x-0.6\rangle^1 - \frac{1.5}{2}\langle x-0.6\rangle^2 + \frac{1.5}{2}\langle x-1.8\rangle^2 - \overset{1.44}{\cancel{M_0}}\langle x-2.6\rangle^0$$

We have one single expression for entire beam.

$$M(x) = EI\frac{d^2y}{dx^2}$$

$$EI\frac{dy}{dx} = \frac{2.6}{2}\langle x-0\rangle^2 - \frac{1.2}{2}\langle x-0.6\rangle^2 - \frac{1.5}{6}\langle x-0.6\rangle^3 + \frac{1.5}{6}\langle x-1.8\rangle^3 - 1.44\langle x-2.6\rangle^1 + C_1$$

$$EIy = \frac{2.6}{6}\langle x-0\rangle^3 - \frac{1.2}{6}\langle x-0.6\rangle^3 - \frac{1.5}{24}\langle x-0.6\rangle^4 + \frac{1.5}{24}\langle x-1.8\rangle^4 - \frac{1.44}{2}\langle x-2.6\rangle^2 + C_1x + C_2$$

From boundary condition
at $x=0$, $y=0$ at $x=L$, $y=0$

at $x=0$, $y=0 \Rightarrow C_2=0$.
at $x=L$, $x=3.6$, $y=0$

$$0 = \frac{2.6}{6} \times 3.6^3 - \frac{1.2}{6} \times 3^3 - \frac{1.5}{24} \times 3^4 + \frac{1.5}{24} \times 1.8^4 - \frac{1.44(1)^2}{2} + C_1 \times 3.6$$

$$= 20.2176 - 5.4 - 5.0625 + 0.6561 - 6.72 + C_1 \times 3.6$$

$$C_1 = -2.692$$

We need deflection at D, distance = 1.8m from A.

$$EIy = \frac{2.6}{6} \times 1.8^3 - \frac{1.2}{2} \times 1.2^2 - \frac{1.5}{6} \times 1.2^4 + 0 - 0 - 2.692 \times 1.8$$

$$EIy = 2.5272 - 0.864 - 0.432 - 4.8456$$

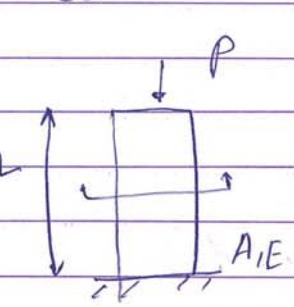
$$EIy = \frac{2.6}{6} \times 1.8^3 - \frac{1.2}{6} \times (1.2)^3 - \frac{1.5}{24} \times 1.2^4 - 2.692 \times 1.8$$

$$= 2.5272 - 0.3456 - 0.1296 - 4.8456$$

$$y = \frac{-2.7936}{EI}$$

(82) Column Buckling

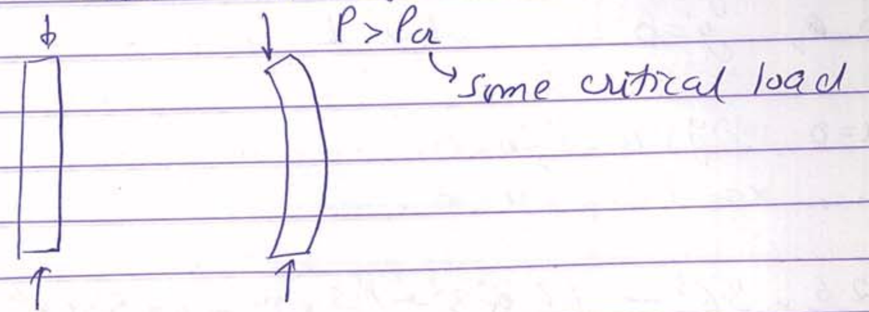
Columns - what we know till now

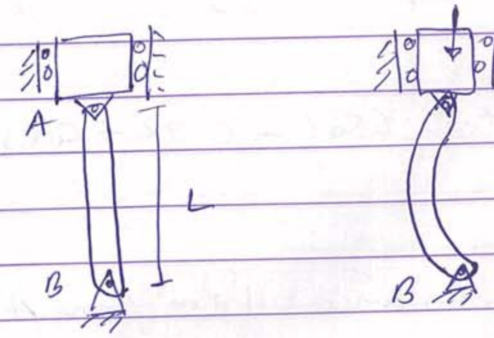


- Axially loaded members
  Stress $\sigma = P/A$, Deformation $\delta = PL/AE$
- We marked such members safe when
  $\sigma < \sigma_{allowable}$, $\delta < \delta_{allowable}$

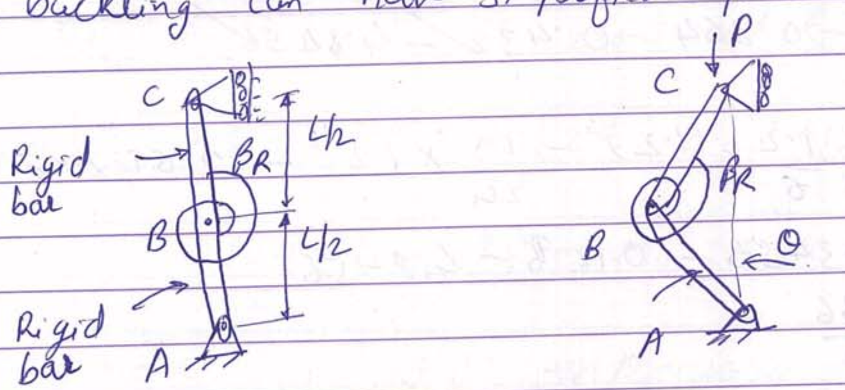
Buckling is a kind of uncertainity that occur due to eccentric loading or non-uniform material.

· We did not account for this →



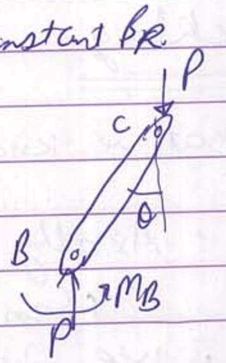


Buckling of column leads to instability. Above buckling can have simplified equivalence as →



Rigid rods AB & BC connected at B.
Torsional spring at B with spring constant β_R.

If we draw FBD for BP



Due to P we are having a disturbing moment

$$\text{Disturbing moment} = P\left(\frac{L}{2}\sin\theta\right) = M_D$$

Restoring / Balancing moment = $\beta_R \times 2\theta = M_B$

For structure to be stable, $M_D = M_B$, just before reaching limiting point

if $M_D > M_B$, then there is unstable equilibrium.

if $M_D < M_B$, then the system tends to return to original system is stable.

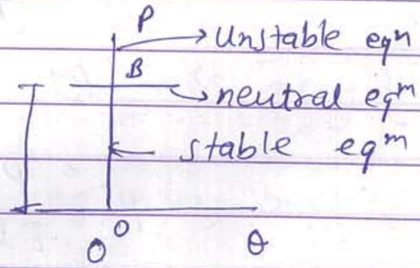
if $M_D = M_B$. For some load critical load $P = P_{cr}$, both couples are equal. System is neutral.

We equate M_D & M_B.

$$M_B - P\left(\frac{\theta L}{2}\right) = 0$$

$$\left(2\beta_r - \frac{PL}{2}\right)\theta = 0$$

$$\boxed{P = P_{cr} = \frac{4\beta r}{L}}$$



β_r = rotation stiffness of spring

Now, if $P > P_{cr}$ → unstable eqm

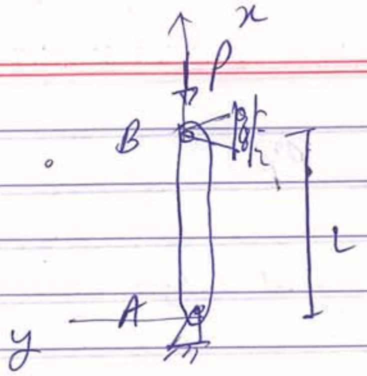
if $P < P_{cr}$ → stable eqm

if $P = P_{cr}$ → critical load point.

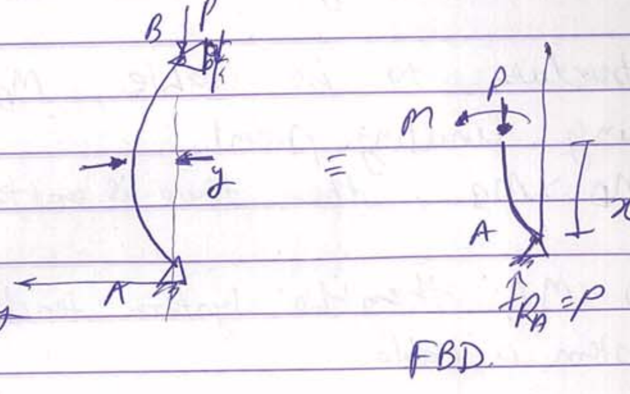
The eqn $P_{cr} = \frac{4\beta r}{L}$ is independent of θ.

(83) Euler Bucking formula: Pin ended columns

- The formula we derived in earlier slides for pin-pin ended columns were using rigid elements & rotational springs → Analogically similar, but not ~~exat~~ exact.
- Exact expression for critical load P_{cr} given by euler



Pin ended column



Expression for FBD

$$M = -Py$$

We know $EI \frac{d^2y}{dx^2} = M = -Py$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI} y = 0$$

$$p^2 = \frac{P}{EI}$$

$$\Rightarrow \frac{d^2y}{dx^2} + p^2 y = 0 \quad \text{— (1)}$$

We have a general solution for this →

$$y = C_1 \sin(px) + C_2 \cos(px)$$

Boundary conditions:-

$$x = 0, y = 0, \quad x = L, y = 0$$

$$C_2 = 0 \; ; \; C_1 \sin(pL) = 0$$

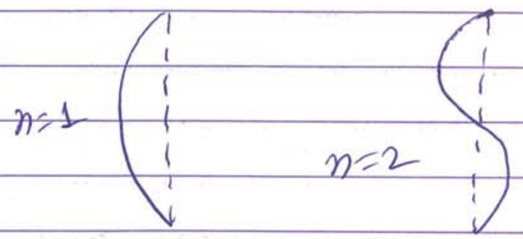
As, C_1 can't be 0 as it will mean no buckling,
So, we equate $\sin pL = 0$

$$pL = n\pi \qquad \cancel{\frac{PL}{EI}} = n\pi \; , n = 1, 2, 3, \ldots$$

$$p^2 = \frac{P}{EI} \Rightarrow \boxed{P = \frac{n^2 \pi^2 EI}{L^2}} \; ; \; n = 1, 2, 3, \ldots$$

So, value of $\boxed{y = C_1 \sin\left(\frac{n\pi x}{L}\right)}$.

P & y are interdependent., n represent modeshapes.



- Different values of n leads to different 'mode shapes'.

When n=1, the column has already buckled, so physically it is not possible to get n=2.

So, Practically, only n=1 is possible

$$\therefore \quad P = \boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \quad \text{and} \quad \boxed{y = C_1 \sin\left(\frac{\pi x}{L}\right)}$$

Euler Buckling load

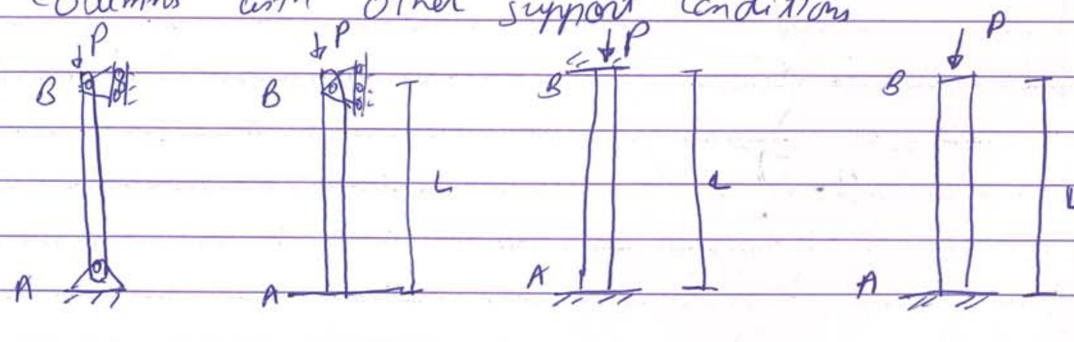
Corresponding buckled shape

C_1 remains undetermined

(84) Columns with other end-Support conditions

The buckling depends upon the axis that we are trying to buckle.

Columns with other support conditions



Pin-Pin Pin-Fixed Fixed-fixed Free-fixed

All other support conditions fall back to pin-pin condition. So, first we studied pin-pin condition.

Concept of effective length (Le)

- Extension of Euler's formula for pin-ended columns to establish equivalence.

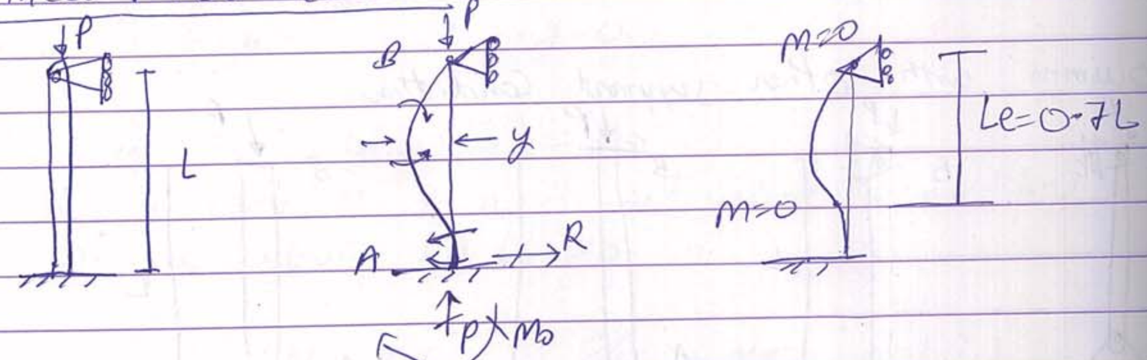
- Effective length Le of a column with any support condition is the length of an equivalent pin-pin classic Euler column.

- Note that in a pin-pin column, moment at the ends are 0.

- So, easiest way to remember: Le for a column with given support conditions is the distance b/w points of inflection / contraflexure where BM = 0

$$P_{cr} = \frac{\pi^2 EI}{Le^2}$$

Pinned-fixed column

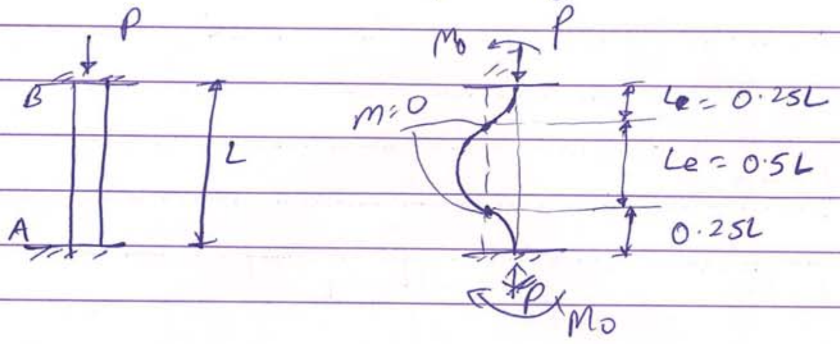


We go from sagging to hogging. So there is a point in b/w where moment is 0.

$$P_{cr} = \frac{\pi^2 EI}{(Le)^2} = \frac{\pi^2 EI}{(0.7L)^2} = \frac{2.05\pi^2 EI}{L^2}$$

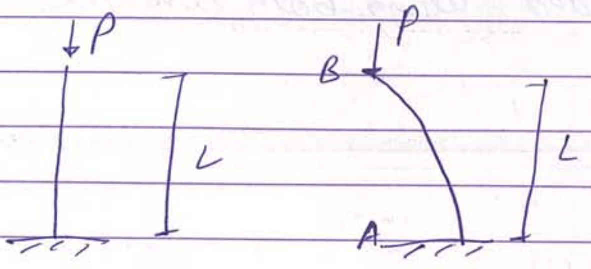
- Fixed - Fixed column

It is only free to come down, rest all are restricted at location B in fixed fixed column



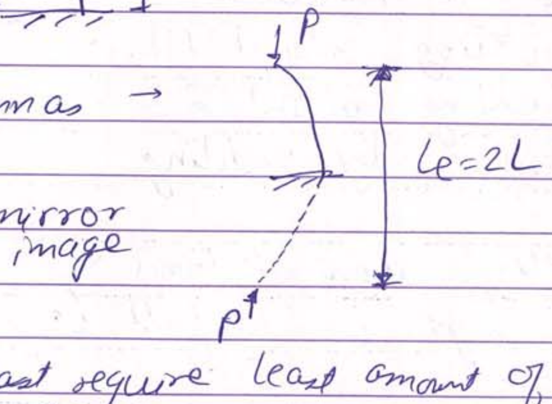
$$P_{cr} = \frac{\pi^2 EI}{(Le)^2} = \frac{\pi^2 EI}{0.25L} = \boxed{\frac{4\pi^2 EI}{L^2} = P_{cr}}$$

- Free fixed column



Except at B bending moment is not 0.

If we represent our column as →



This type of column has least require least amount of force for buckling. $P_{cr} = \frac{\pi^2 EI}{Le^2} = \frac{\pi^2 EI}{4L^2} =$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

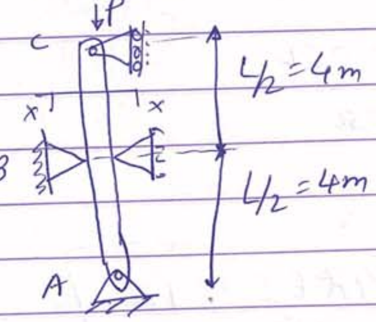
Summary

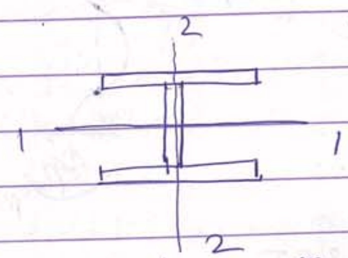
(a) Pinned-Pinned (b) Fixed-free (c) Fixed-fixed (d) Fixed-pin

$P_{cr} = \frac{\pi^2 EI}{L^2}$ $P_{cr} = \frac{\pi^2 EI}{4L^2}$ $P_{cr} = \frac{4\pi^2 EI}{L^2}$ $P_{cr} = \frac{2.05\pi^2 EI}{L^2}$

$l_{eff} = L$ $L_e = 2L$ $L_e = L/2$ $L_e = 0.7L$

(85) Example





Section x-x

Consider $E = 200\,GPa$, $\sigma_y = 300\,MPa$

$I_{1-1} = 3060\,cm^4$, $I_{2-2} = 162\,cm^4$, $A = 39.5\,cm^2$

- Pin at A & C provided along both directions (1-1 and 2-2)
- Pin at B only along 1-1

⇒ Need to calculate 3 things:

① Buckling about 1-1 axis
② Buckling about 2-2 axis
③ Failure by yielding

① Buckling about 1-1 axis

Our length will be full length ~~tef~~ $L_{eff} = 4+4 = 8m$.

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times (200\,GPa) \times (3060\,cm^4)}{8^2}$$

$$= 9444\,KN$$

(2) Buckling about 2-2 axis

$$P_{cr} = \frac{\pi^2 E I_{2-2}}{L_e^2}$$

Here our Leffective will be 4 m. As there is pin at location B. which will restrict its motion.
So, $l_{eff} = 4m$

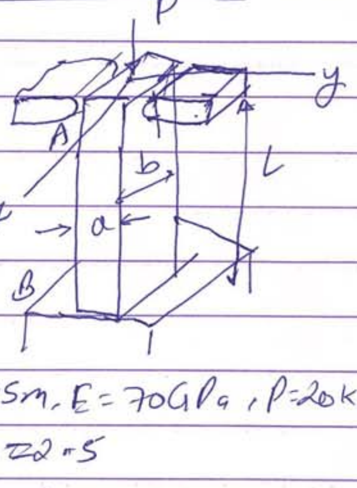
$$P_{cr} = \frac{\pi^2 E I}{4^2} = \frac{\pi^2 \times (200 GPa) \times 162 \, cm^4}{4^2} = 200 \, kN.$$

(3) Failure by yielding

Yield Load $= \sigma_y \times A = 1185 \, kN.$

Max allowable load $= \frac{\min(944, 200, 1185)}{3} = \frac{200}{3} = 66.66 \, kN.$
(factor of safety)

(86) Example - 2



$L = 0.5 m$, $E = 70 GPa$, $P = 20 kN$
FOS $= 2.5$

Two smooth & rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry but allows it to move in the other plane.
a) Determine ratio a/b of the two sides of the cross-section corresponding to the most efficient design against buckling
(b) Design the most efficient cross-section for the column.

a) At B, it is fixed. At A, it is free in z-axis & like a paint joint (due to smooth surface) in the Y-axis.